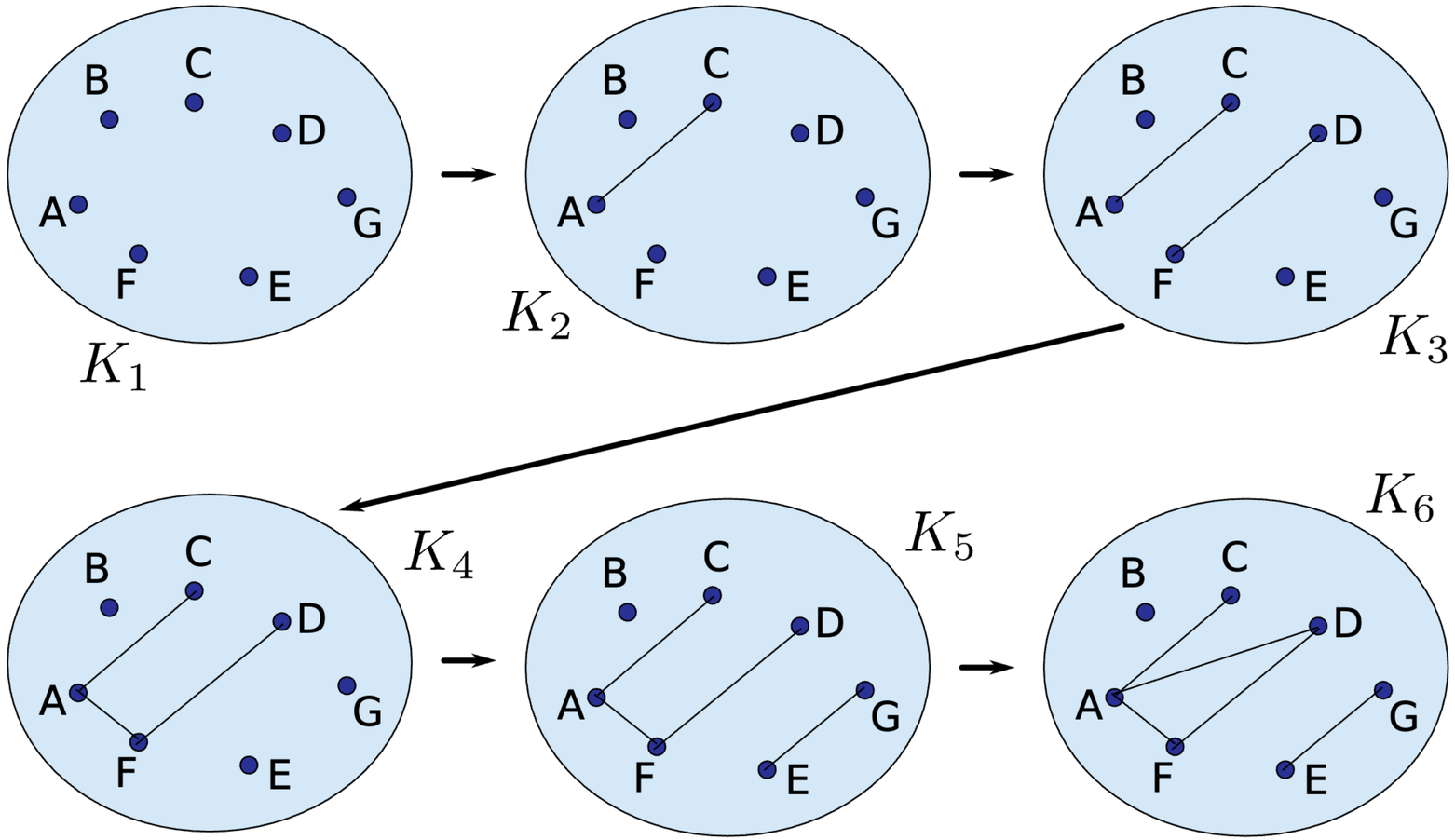
Persistence

Simplicial Complex Filtration

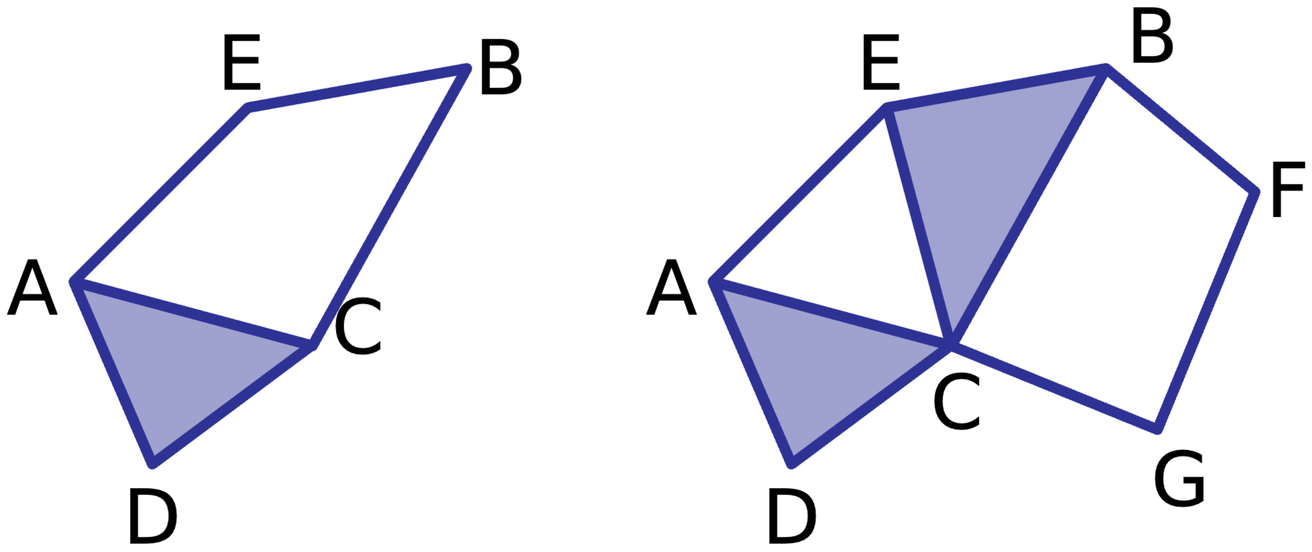
A *filtration* $\mathcal{F} = \mathcal{F}(K)$ of a simplicial complex K is a nested sequence of its subcomplexes $\mathcal{F} : \emptyset = K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n = K.$



Simplicial Maps

A *simplicial map* between abstract simplicial complexes $f : K \to L$ is induced by a map on the vertices $f : V(K) \to V(L)$. For every $\sigma \in K, f(\sigma) \in L$.

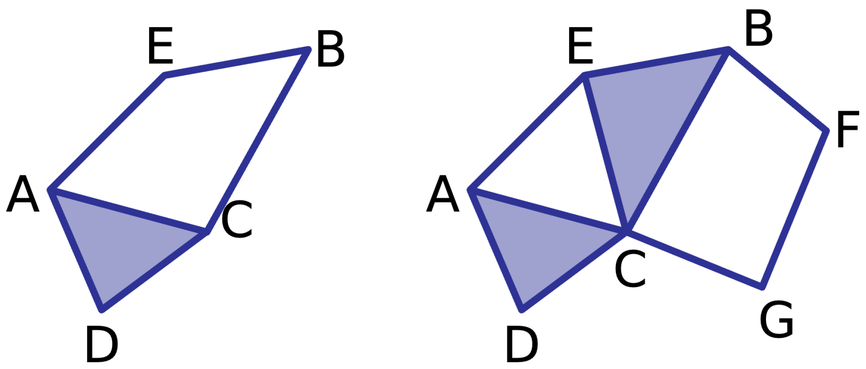
Cheap trick: we can use inclusion!



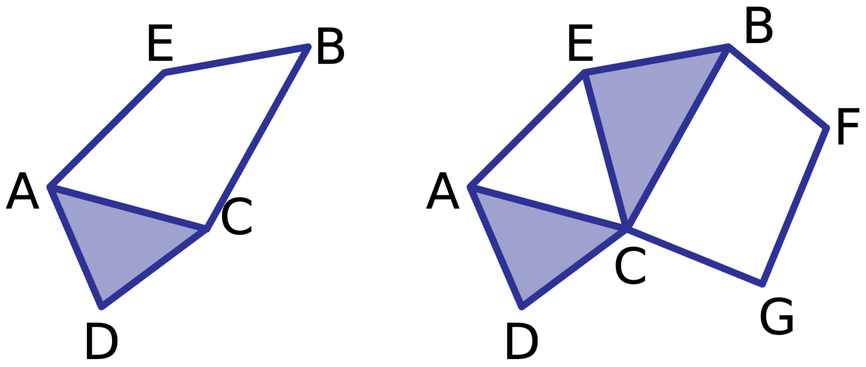
Induced Map

Given a simplicial map $f : K \to L$, the induced map on homology is defined by

$$f_* : H_p(K) \longrightarrow H_p(L)$$
$$[\alpha] \longmapsto [f_\#(\alpha)]$$



Commutative Diagram



A diagram commutes if any path of functions is equal.

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & C_2(K) & \xrightarrow{\partial_2^K} & C_1(K) & \xrightarrow{\partial_1^K} & C_0(K) & \xrightarrow{\partial_0^K} & 0 \\ & & \downarrow f_\# & & \downarrow f_\# & & \downarrow f_\# & & \\ \cdots & \longrightarrow & C_2(L) & \xrightarrow{\partial_2^L} & C_1(L) & \xrightarrow{\partial_1^L} & C_0(L) & \xrightarrow{\partial_0^L} & 0 \end{array}$$

Claim: $f_\# \circ \partial^K = \partial^L \circ f_\#$.

Example

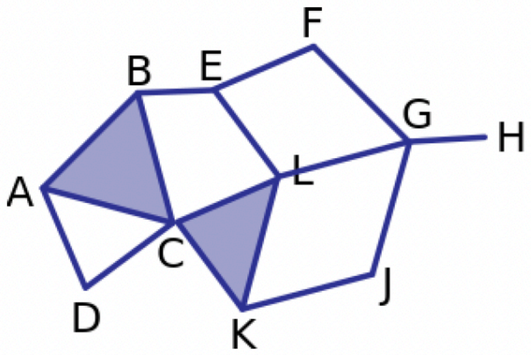
If $H_1(K)$ generated by

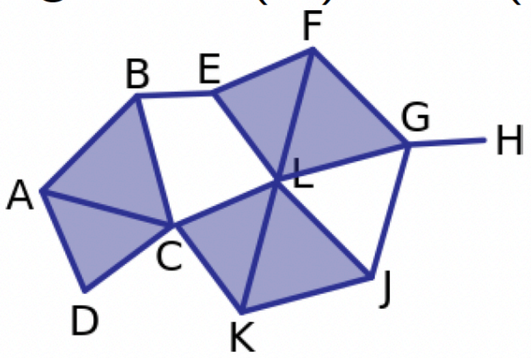
$$[\alpha_1] = [AC + CD + AD]$$
$$[\alpha_2] = [BE + EL + CL + CB]$$
$$[\alpha_3] = [EF + FG + GL + EL]$$
$$[\alpha_4] = [GL + GJ + JK + LK]$$

and $H_1(L)$ generated by

$$[\beta_1] = [BE + EL + CL + CB]$$
$$[\beta_2] = [GL + GJ + LJ]$$

write the matrix representing $f_* : H_1(K) \to H_1(L)$ induced by inclusion.





Try it

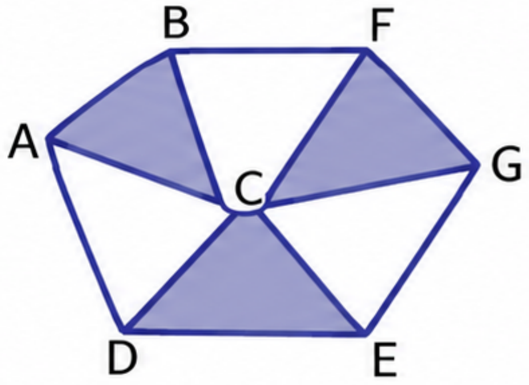
If $H_1(K)$ is generated by

$$[\alpha_1] = [AB + BC + CA]$$
$$[\alpha_2] = [CD + DE + EC]$$
$$[\alpha_3] = [BF + FG + GB]$$

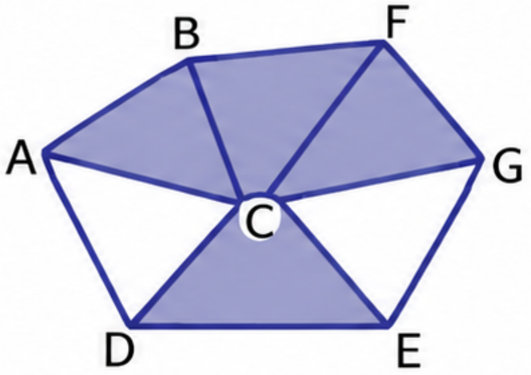
and $H_1(L)$ is generated by

$$[\beta_1] = [AB + BF + FA]$$
$$[\beta_2] = [CD + DE + EC]$$

write the matrix representing $f_* : H_1(K) \to H_1(L)$ induced by inclusion.



K



L

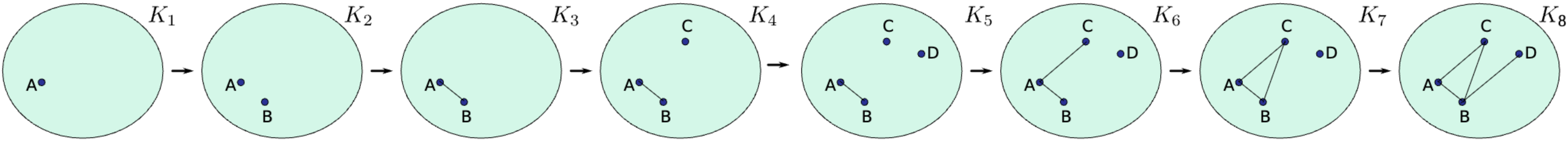
Persistence Module

Given a filtration, the p-dimensional *persistence module* is

$$H_p(\mathcal{F}) : 0 = H_p(K_0) \to H_p(K_1) \to H_p(K_2) \to \cdots \to H_p(K_n) = H_p(K)$$

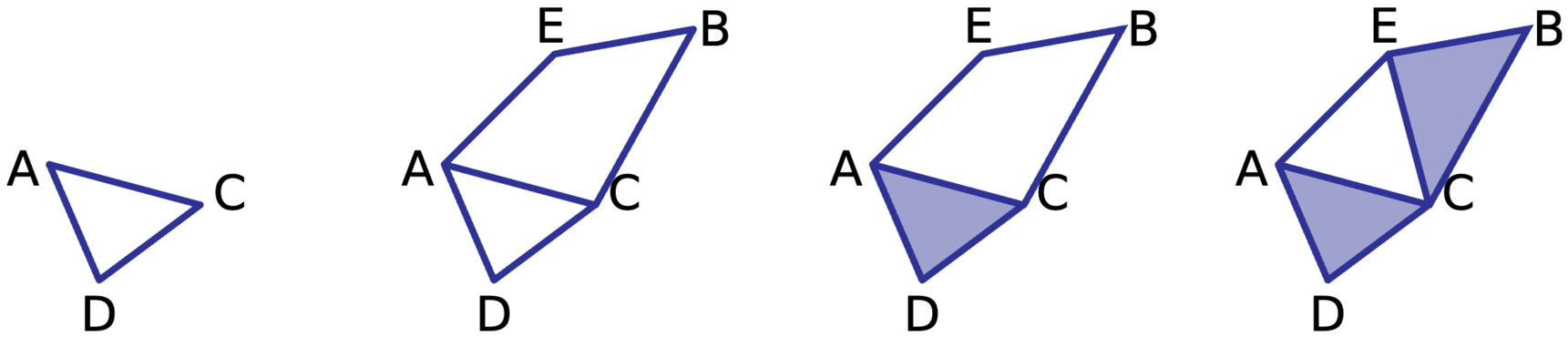
with maps $h_p^{i,j} : H_p(K_i) \to H_p(K_j)$ induced by inclusion

What's going on algebraically?



$$H_0(K_1) \longrightarrow H_0(K_2) \longrightarrow H_0(K_3) \longrightarrow H_0(K_4) \longrightarrow H_0(K_5) \longrightarrow H_0(K_6) \longrightarrow H_0(K_7) \longrightarrow H_0(K_8)$$

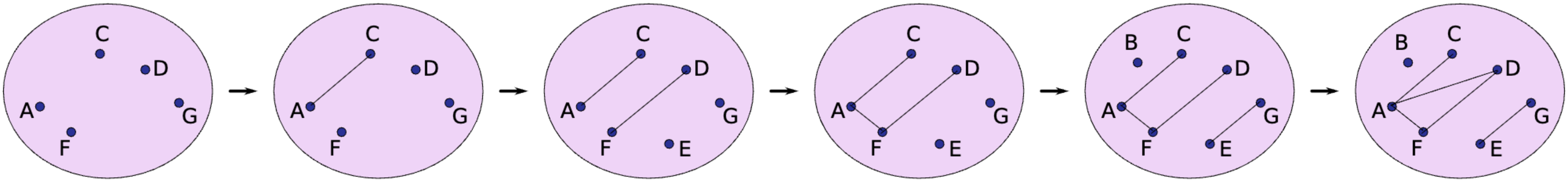
Another Example



K_0 K_1 K_2 K_3

$$H_1(K_0) \xrightarrow{h_1^{0,1}} H_1(K_1) \xrightarrow{h_1^{1,2}} H_1(K_2) \xrightarrow{h_1^{2,3}} H_1(K_3)$$

Try it: Where do the generators go?



K_1 K_2 K_3 K_4 K_5 K_6

$$H_0(K_1) \longrightarrow H_0(K_2) \longrightarrow H_0(K_3) \longrightarrow H_0(K_4) \longrightarrow H_0(K_5) \longrightarrow H_0(K_6)$$

The pth Persistent Homology Group

$$H_p(K_0) \to H_p(K_1) \to H_p(K_2) \to \cdots \to H_p(K_i) \to \cdots \to H_p(K_j) \to \cdots \to H_p(K_n)$$

The *pth persistent homology groups* are the images induced by inclusion

$$H_p^{i,j} = \operatorname{Im}\left(H_p(K_i) \to H_p(K_j)\right).$$

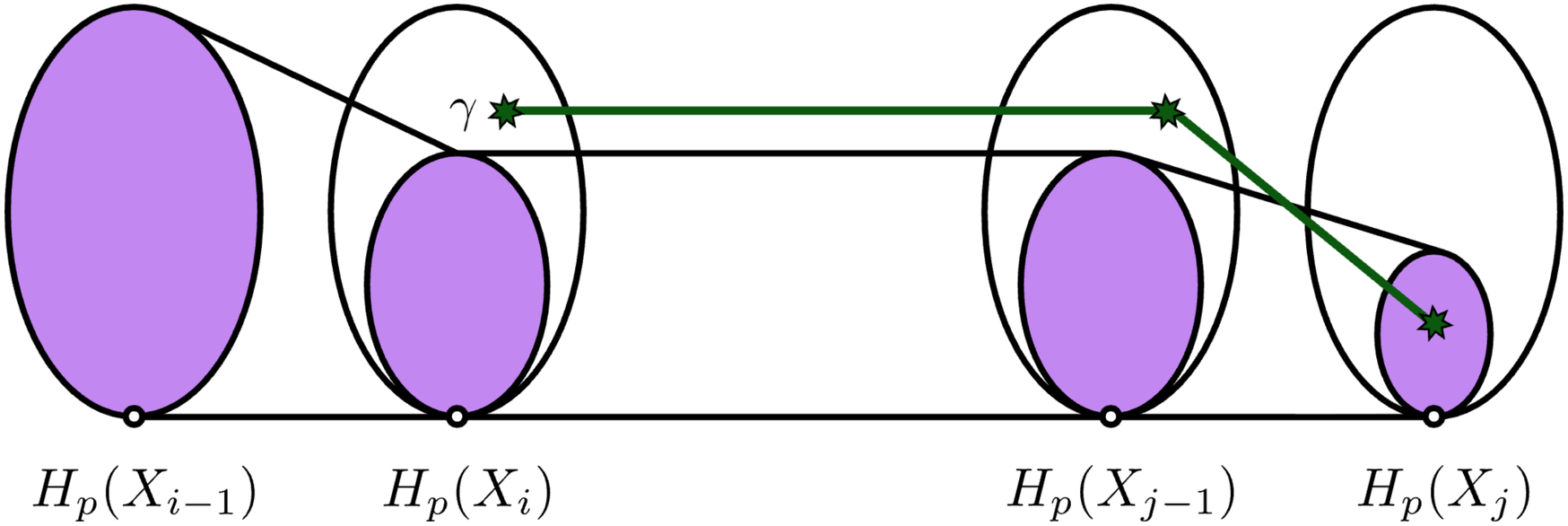
The pth persistent Betti numbers are the ranks

$$\beta_p^{i,j} = \operatorname{rank} H_p^{i,j}$$

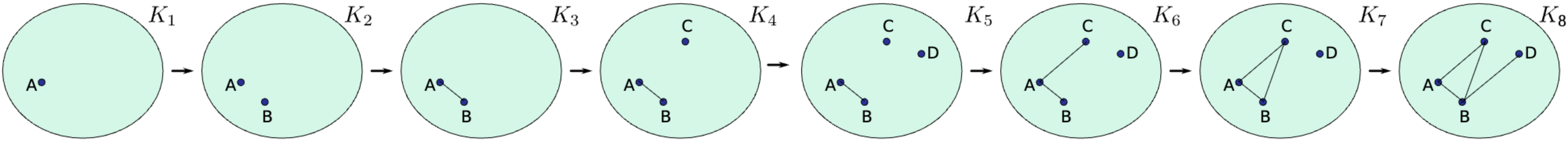
Birth and Death

Birth: A class $\gamma \in H_p(K_i)$ is born at K_i if it is not in $H_P^{i-1,i}$

Death: The class γ dies entering K_j if merges with an older class. Specifically, if $h_p^{i,\,j-1}(\gamma) \notin H_p^{i-1,\,j-1}$ but $h_p^{i,\,j}(\gamma) \in H_p^{i-1,\,j}$.



When are generators born and die?

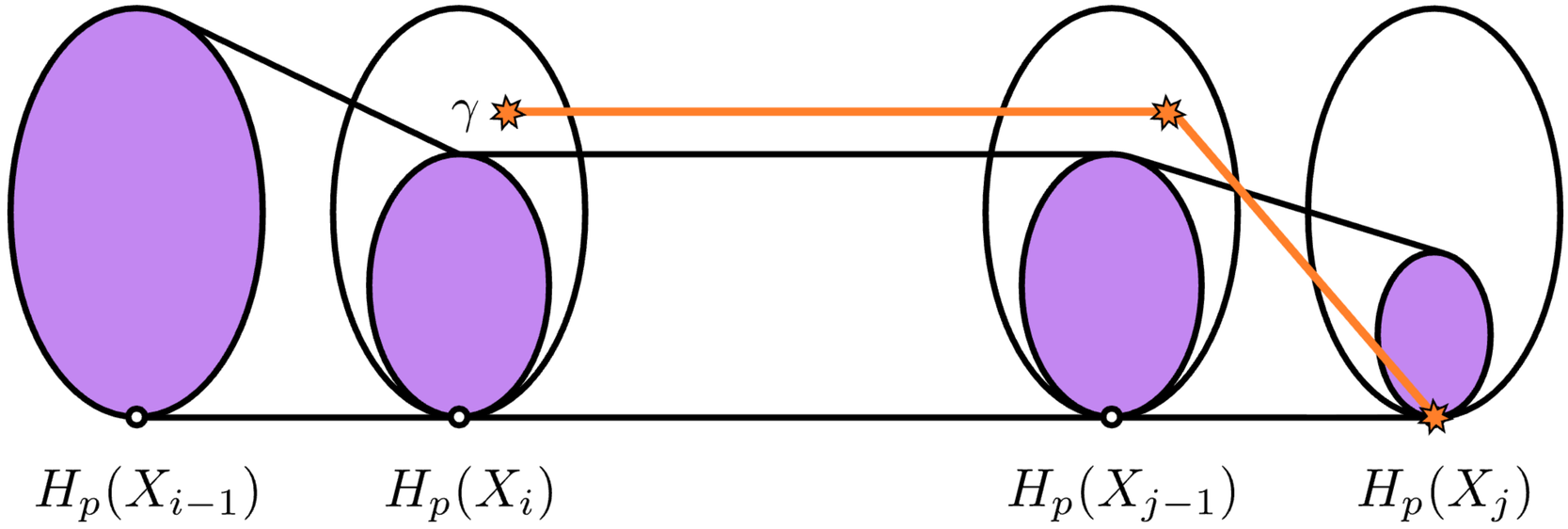


$$H_0(K_1) \longrightarrow H_0(K_2) \longrightarrow H_0(K_3) \longrightarrow H_0(K_4) \longrightarrow H_0(K_5) \longrightarrow H_0(K_6) \longrightarrow H_0(K_7) \longrightarrow H_0(K_8)$$

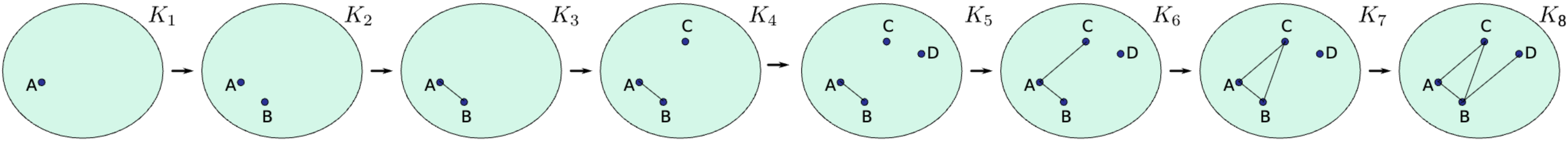
Birth and Death (Book Version)

Birth: A class $\gamma \in H_p(K_i)$ is born at K_i if it is not in $H_P^{i-1,i}$

Death: The class γ dies entering K_j if $\gamma \in H_p(X_{j-1})$ is not trivial, but $h_p^{i-1,\,j}(\gamma) = 0$.

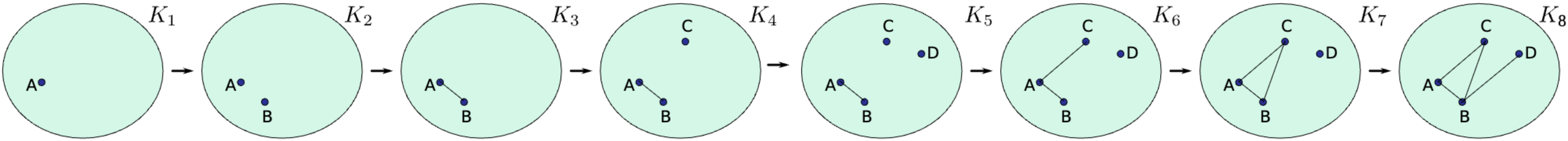


Now where do the elements go?



$$H_0(K_1) \longrightarrow H_0(K_2) \longrightarrow H_0(K_3) \longrightarrow H_0(K_4) \longrightarrow H_0(K_5) \longrightarrow H_0(K_6) \longrightarrow H_0(K_7) \longrightarrow H_0(K_8)$$

Choosing a different basis



$$H_0(K_1) \longrightarrow H_0(K_2) \longrightarrow H_0(K_3) \longrightarrow H_0(K_4) \longrightarrow H_0(K_5) \longrightarrow H_0(K_6) \longrightarrow H_0(K_7) \longrightarrow H_0(K_8)$$

Birth and Death Pairing (Book Version)

Let $[c]$ be a pth homology class that dies (maps to 0) entering X_j. Then, it is born at X_i if and only if there exists a sequence $i_1 \leq i_2 \leq \cdots \leq i_k = i$ for some $k \leq 1$ so that

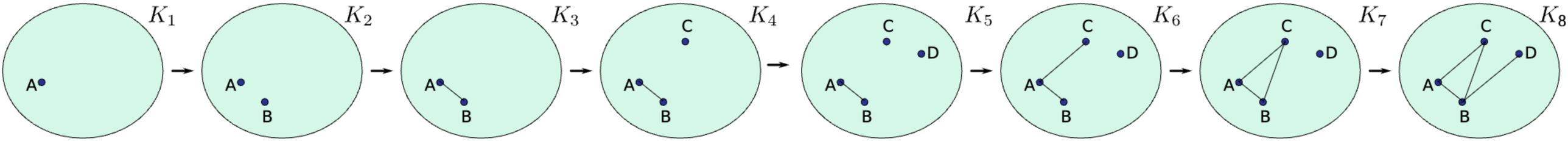
- There is a c_{i_ℓ} where $[c_{i_\ell}]$ is born at X_{i_ℓ} for every $\ell \in \{1, \ \ldots, \ k\}$.
- $[c] = h_p^{i_1,\, j-1}([c_{i_1}]) + \cdots \ + h_p^{i_k,\, j-1}([c_{i_k}])$
- $i_k = i$ is the smallest possible value among any sequences have the above two properties.

Counting Classes

$$0 \to H_p(K_1) \to H_p(K_2) \to H_p(K_3) \to \cdots \to H_p(K_n) \to 0$$

- Attach 0 vector space at the end
- Associate $n+1$ to $a_{n+1} = \infty$
- $\beta_p^{i,j} = \operatorname{rank} H_p^{i,j} = \operatorname{rank}\left(\operatorname{Im}\left(H_p(K_i) \to H_p(K_j)\right)\right)$ counts classes born **at or before** i and dying **after** j

Same example again



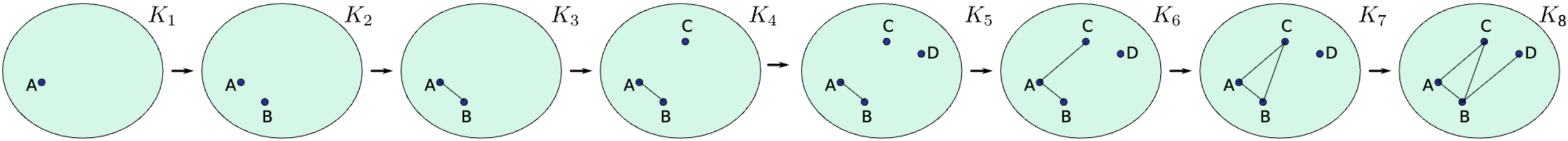
$$H_0(K_1) \longrightarrow H_0(K_2) \longrightarrow H_0(K_3) \longrightarrow H_0(K_4) \longrightarrow H_0(K_5) \longrightarrow H_0(K_6) \longrightarrow H_0(K_7) \longrightarrow H_0(K_8)$$

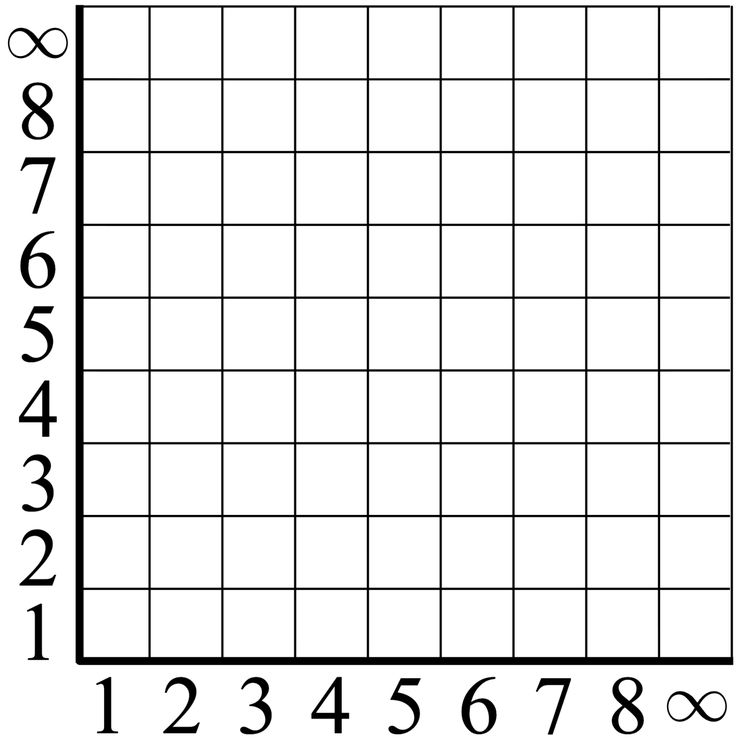
$$[A] \longmapsto [A] \longmapsto [A] \longmapsto [A] \longmapsto [A] \longmapsto [A] \longmapsto [A] \longmapsto [A]$$

$$[A+B] \longmapsto 0 \qquad [A+C] \mapsto [A+C] \longmapsto 0$$

$$[A+D] \mapsto [A+D] \mapsto [A+D] \longmapsto 0$$

Example





Determine $\beta_0^{i,\,j}$ for all pairs $i \leq j$

Persistence Pairing Function

For $0 < i < j \leq n+1$, define

$$\mu_p^{i,j} = \begin{cases} (\beta_p^{i,j-1} - \beta_p^{i,j}) - (\beta_p^{i-1,j-1} - \beta_p^{i-1,j}), & i < j < n+1, \\ \beta_p^{i,j} - \beta_p^{i-1,j}, & i < j = n+1. \end{cases}$$

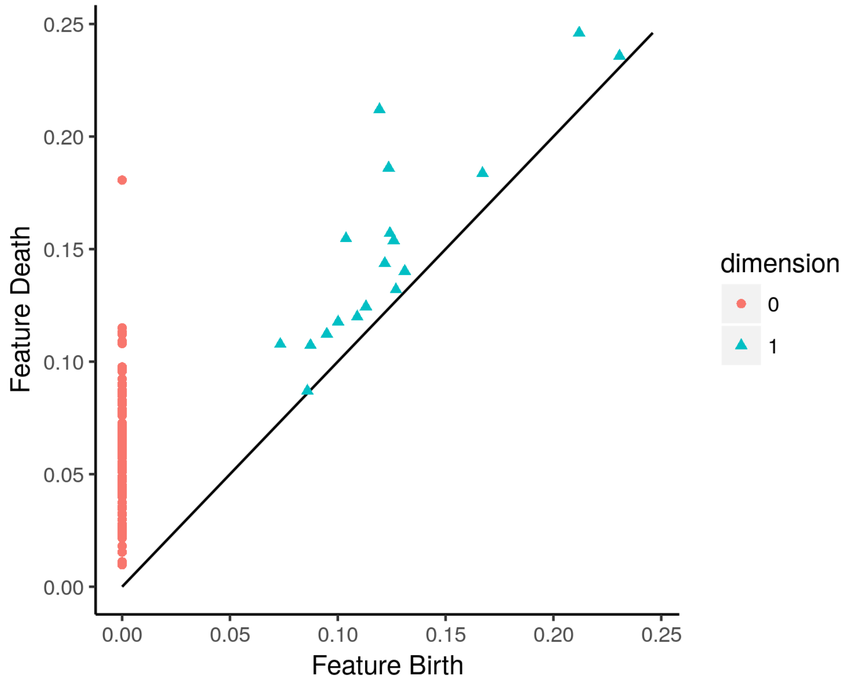
How to get the number of classes born **at** i and dying **at** j ???

Persistence of a Class

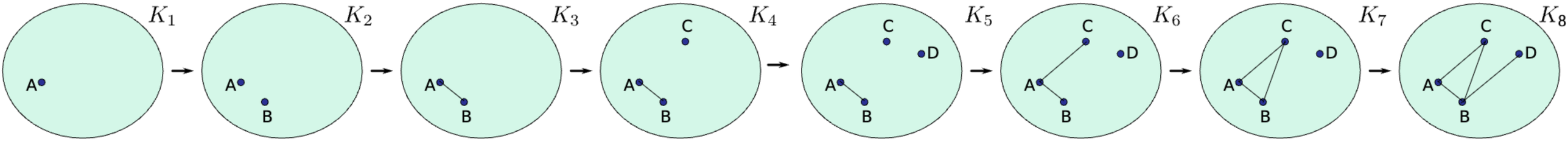
For $\mu_p^{i,\,j} \neq 0$, the persistence Pers$([c])$ of a class $[c]$ that is born at X_i and dies at X_j is defined as Pers$([c]) = a_j - a_i$. When $j = n + 1$ with $a_{n+1} = \infty$, Pers$([c]) = \infty$.

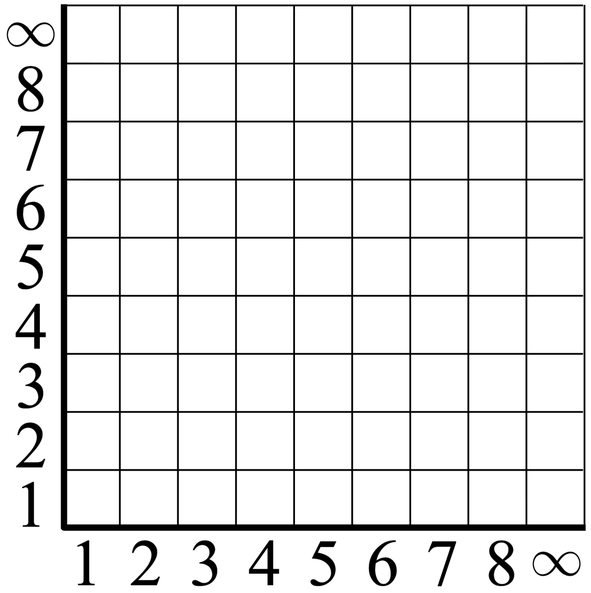
Persistence Diagram

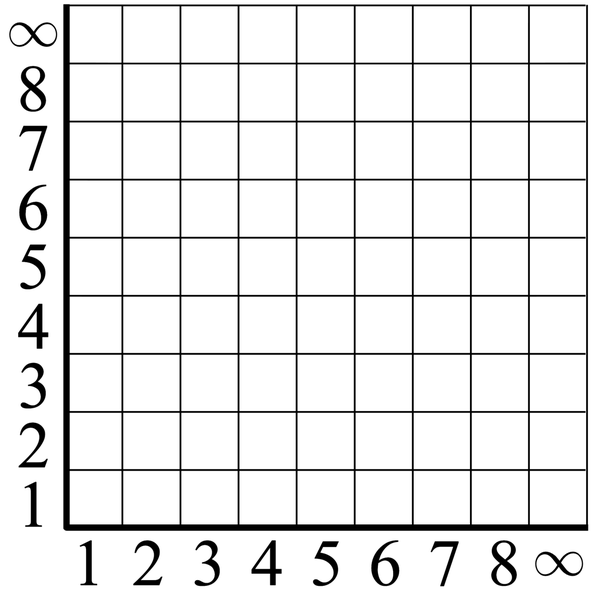
The persistence diagram $\mathrm{Dgm}_p(\mathscr{F})$ (also written $\mathrm{Dgm}_p(f)$) of a filtration induced by a function f is obtained by drawing a point with non-zero multiplicity $\mu_p^{i,\,j}(i < j)$, on the extended plane, where points on the diagonal $\Delta = \{(x,\, x) \in \mathbb{R}^2\}$ are added with infinite pultiplicity.



Example



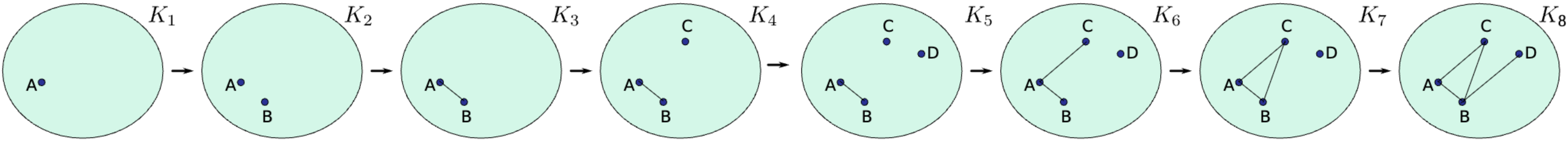




Determine $\mu_0^{i,\,j}$ for all pairs $i \leq j$

$$\mu_p^{i,j} = \begin{cases} (\beta_p^{i,j-1} - \beta_p^{i,j}) - (\beta_p^{i-1,j-1} - \beta_p^{i-1,j}), & i < j < n+1, \\ \beta_p^{i,j} - \beta_p^{i-1,j}, & i < j = n+1. \end{cases}$$

Example



Plot the 0-dimensional persistence diagram