

Filtrations and Induced Map

And maybe persistence?

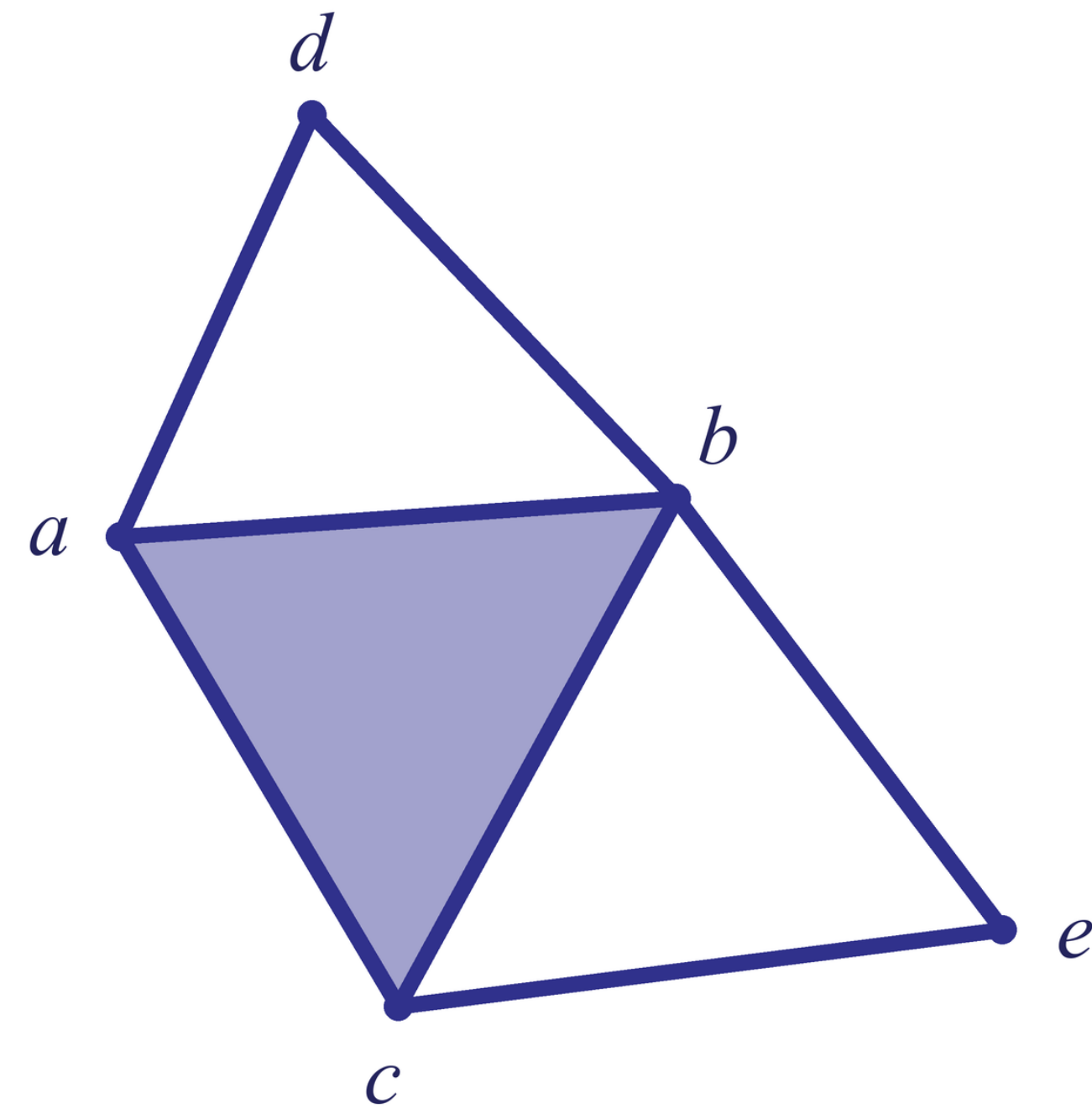
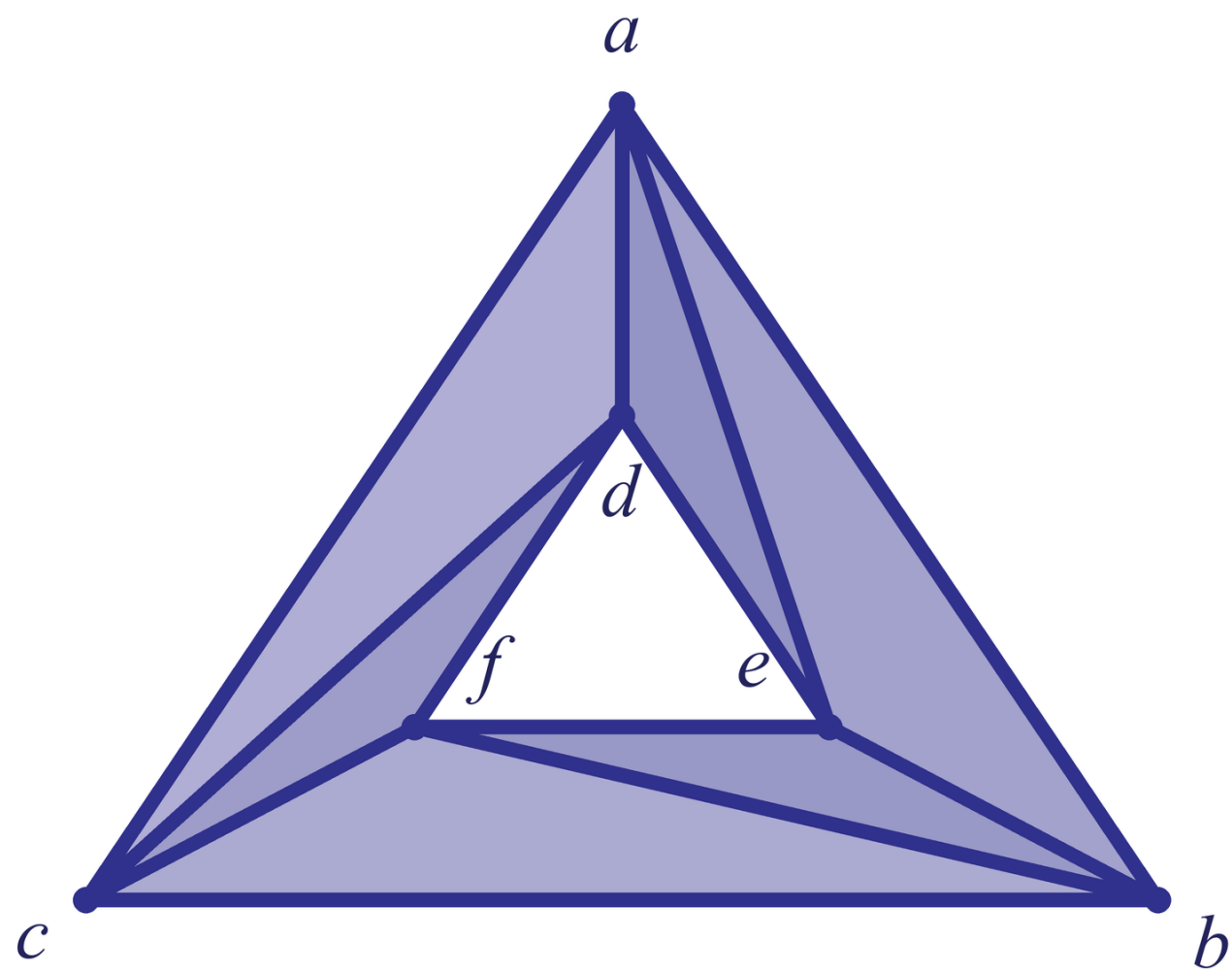
Quick review

- We want to compute the “shape”: we turn “shape” into linear algebra; homology is the machinery
- p -simplex: basis vector
- p -chains: vectors
- Boundary map: matrix
- Cycles: kernel of a matrix
- Boundaries: image of a matrix
- Homology: cycles modulo trivial cycles

Homology!!

The p th homology group is the quotient space

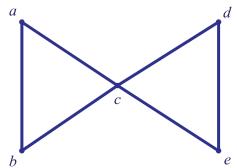
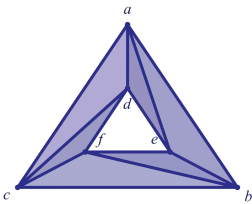
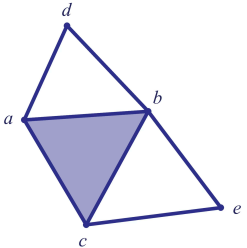
$$H_p(K) := Z_p(K)/B_p(K)$$



Betti Numbers

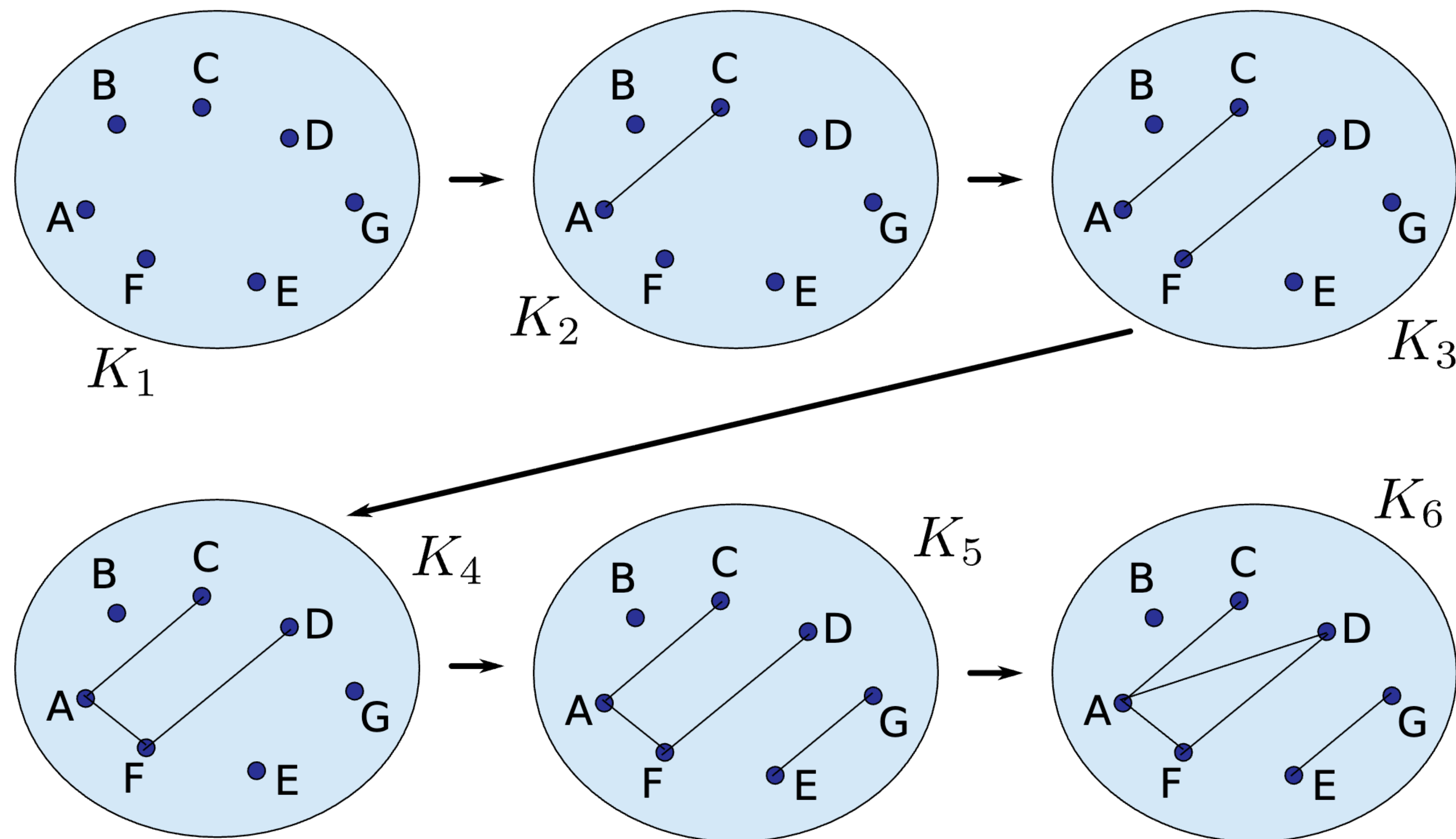
$\beta_k = \dim H_k = \dim Z_k - \dim B_k$

- β_0 : number of connected components
- β_1 : number of independent loop classes
- β_2 : number of independent 2-dimensional homology classes

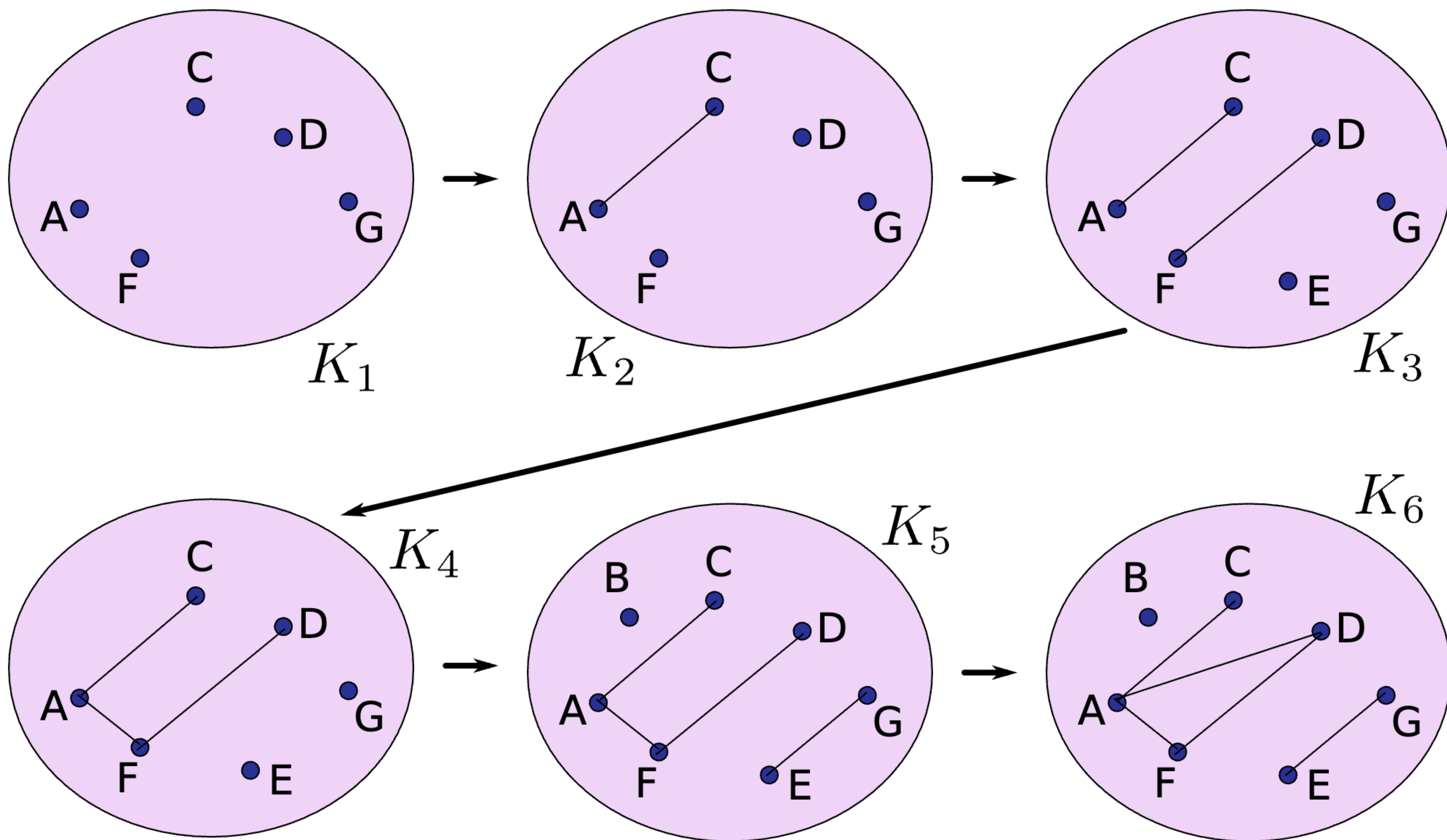
	β_0	β_1	β_2
			
			
			

Simplicial Complex Filtration

A filtration $\mathcal{F} = \mathcal{F}(K)$ of a simplicial complex K is a nested sequence of its subcomplexes $\mathcal{F} : \emptyset = K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n = K$.



Example 2



Monotone Function

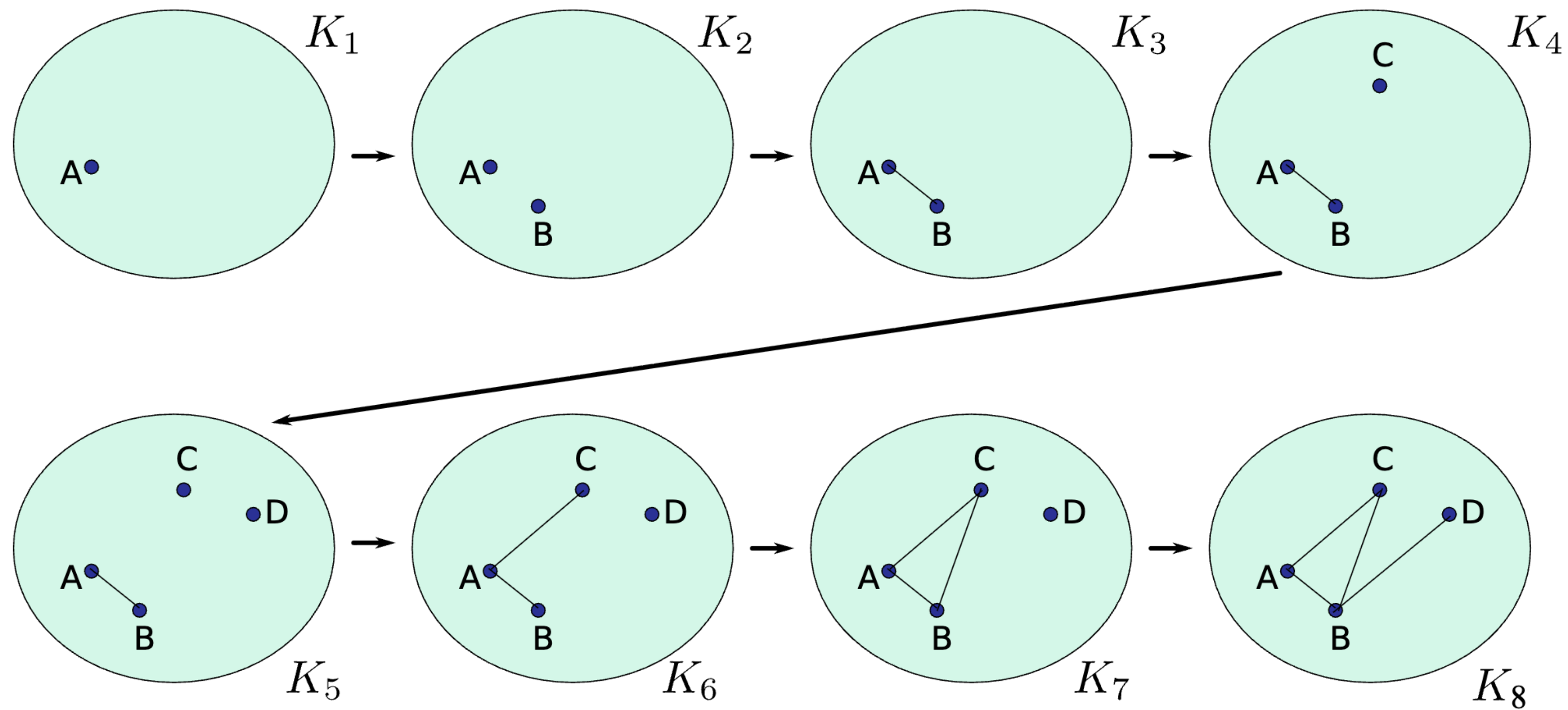
A function $f: K \rightarrow \mathbb{R}$ is called monotone if for every $\tau \leq \sigma, f(\tau) \leq f(\sigma)$.

A simplex-wise filtration means that exactly one simplex is added at each step.

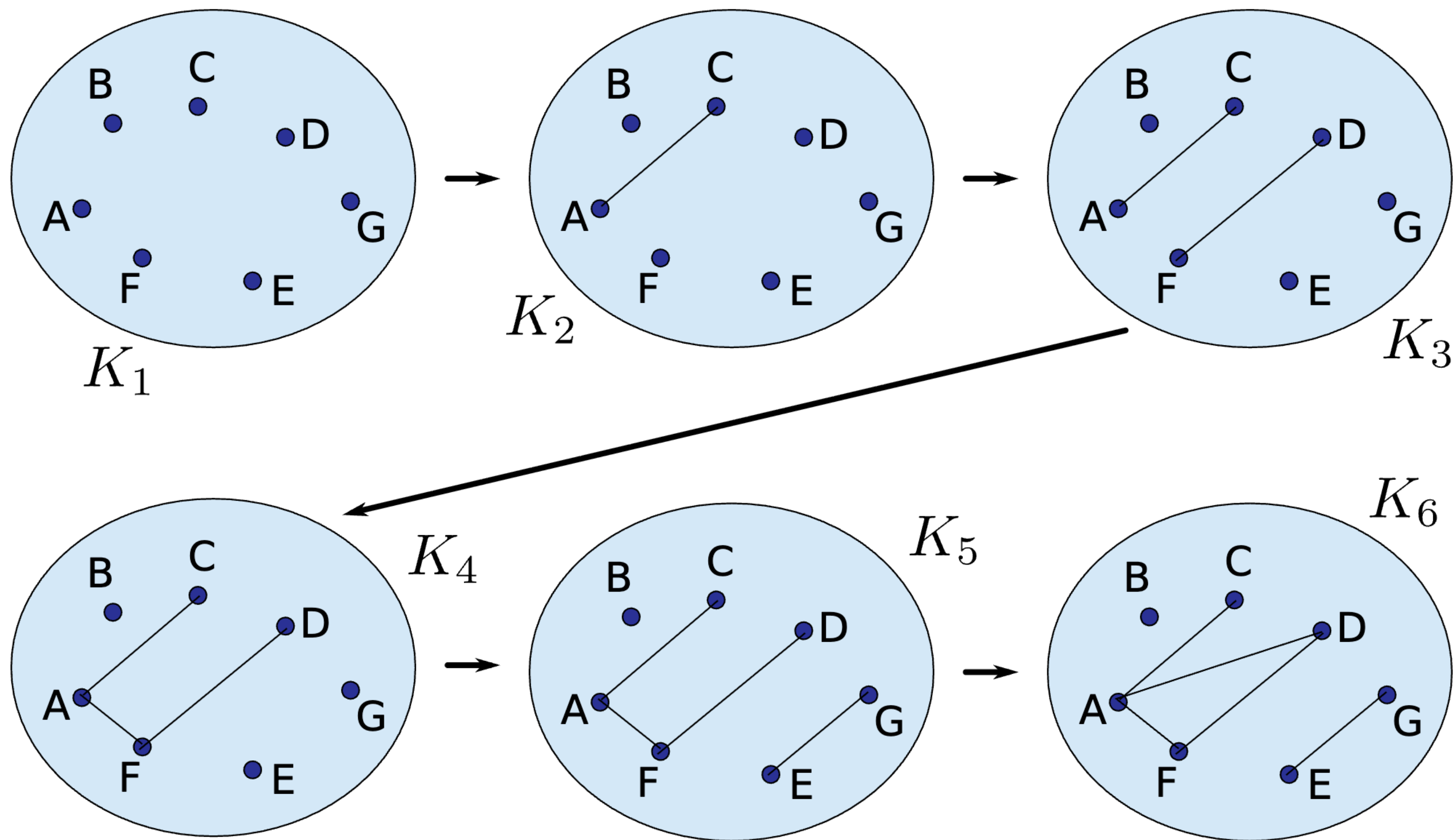
Fix values $a_0 \leq a_1 \leq \cdots \leq a_n$. Let $K_i = f^{-1}(-\infty, a_i]$. The sublevelset filtration induced by this function is defined to be

$$\emptyset = K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n = K.$$

Example: Function inducing filtration

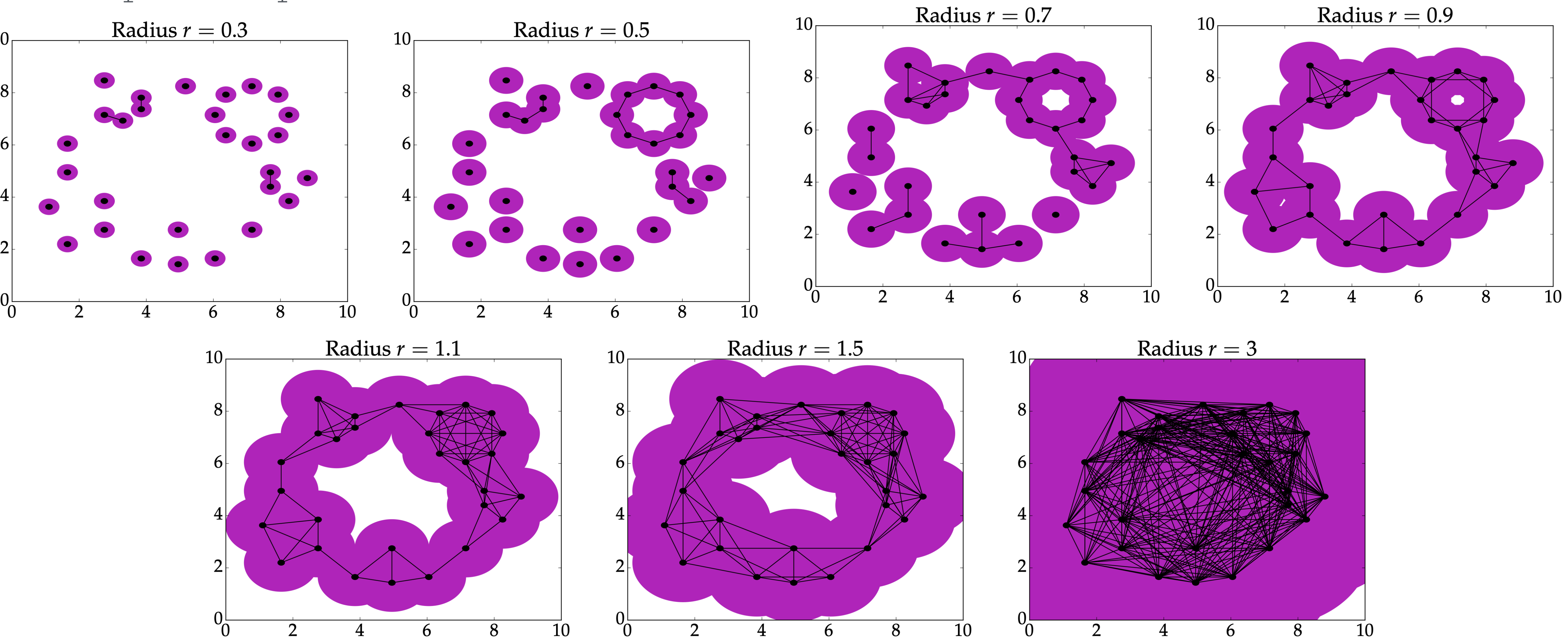


Try it: function inducing filtration



Everything comes from a function

Rips complex filtration



What to do with a filtration?

Betti Curves

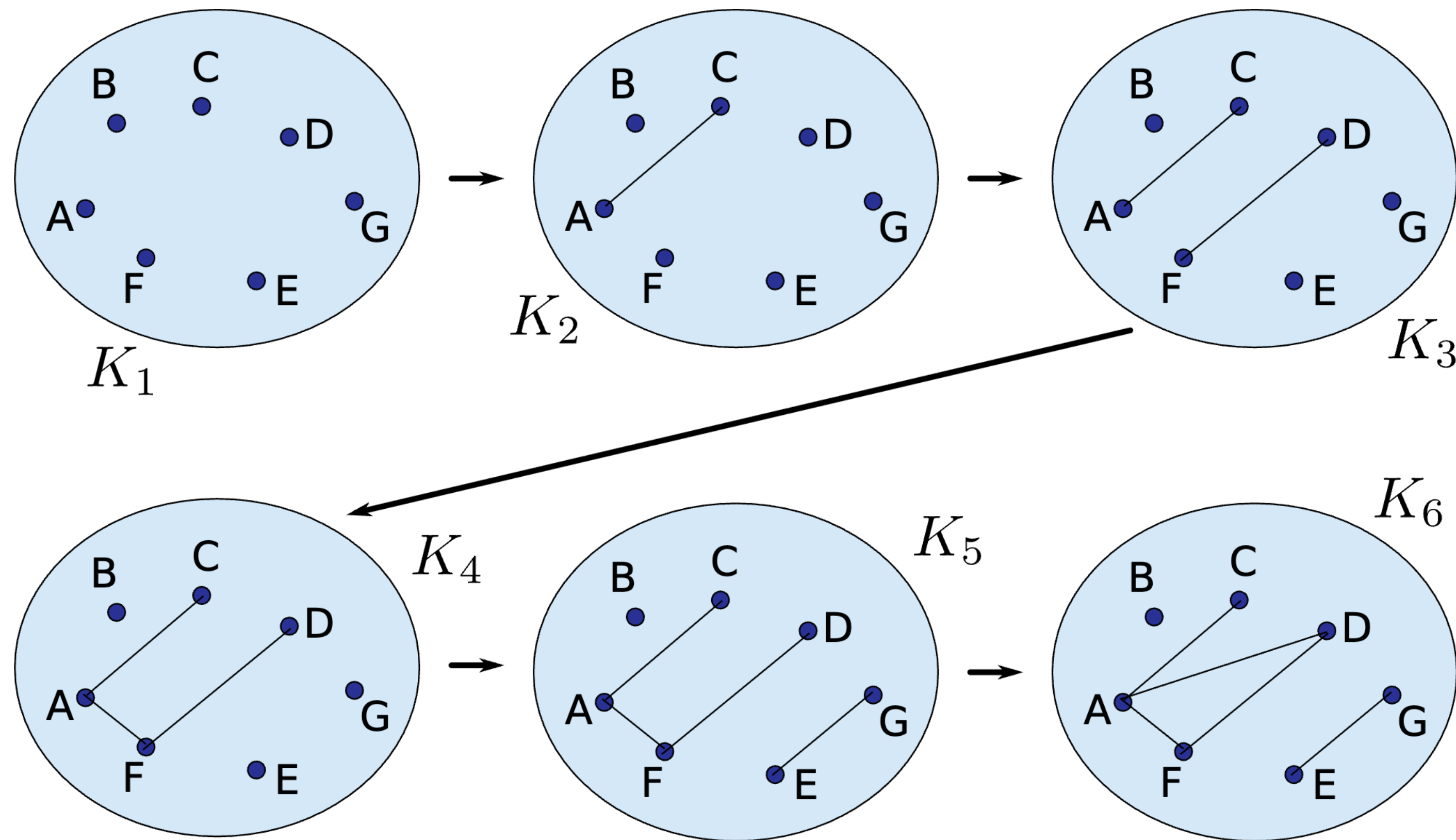
The p -th Betti curve is a function

$$\beta_p(\mathcal{F}) : \mathbb{R} \longrightarrow \mathbb{Z}$$

$$a_i \longmapsto \beta_p(K_i) = \text{rk}(H_p(K_i)).$$

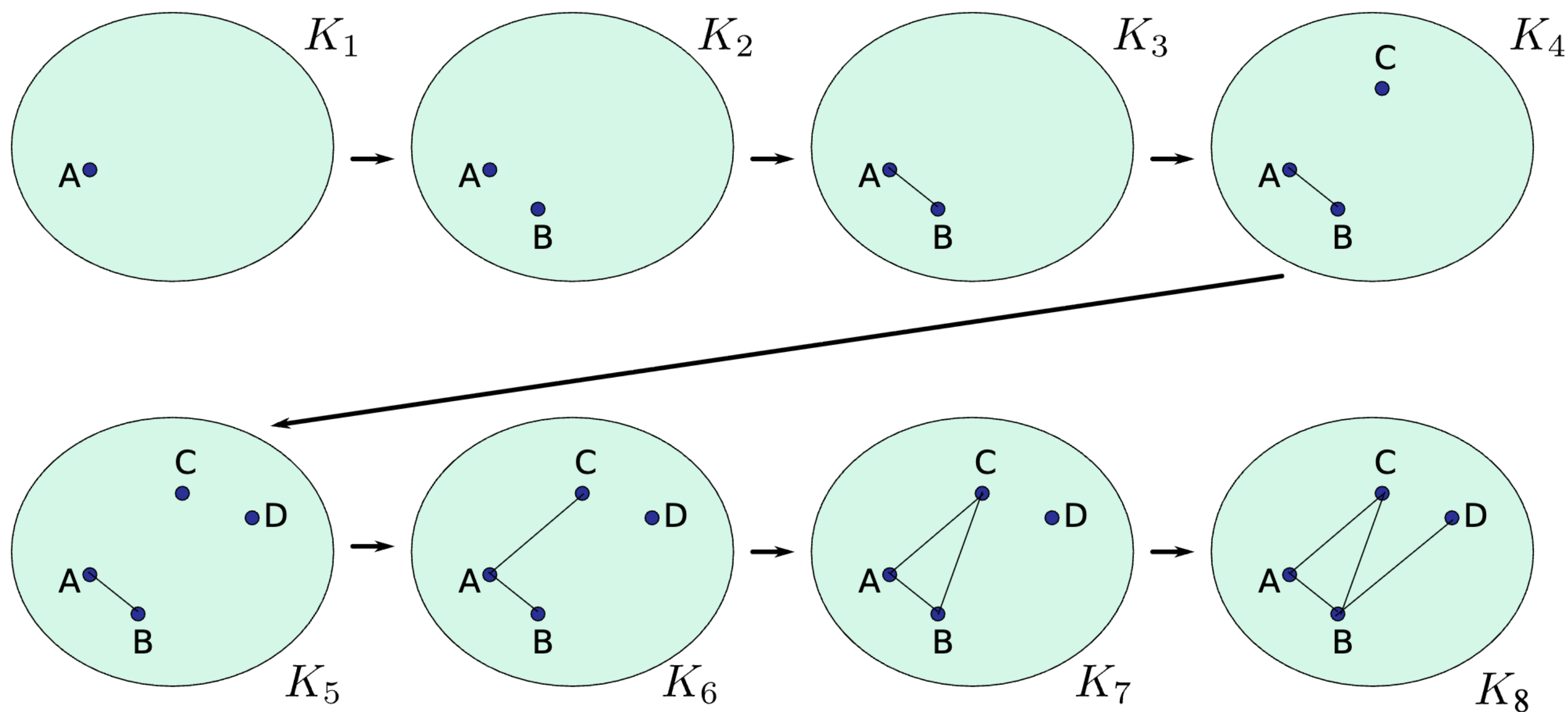
Example

Write the **0**th and **1**st Betti curves for this filtration:



Try it

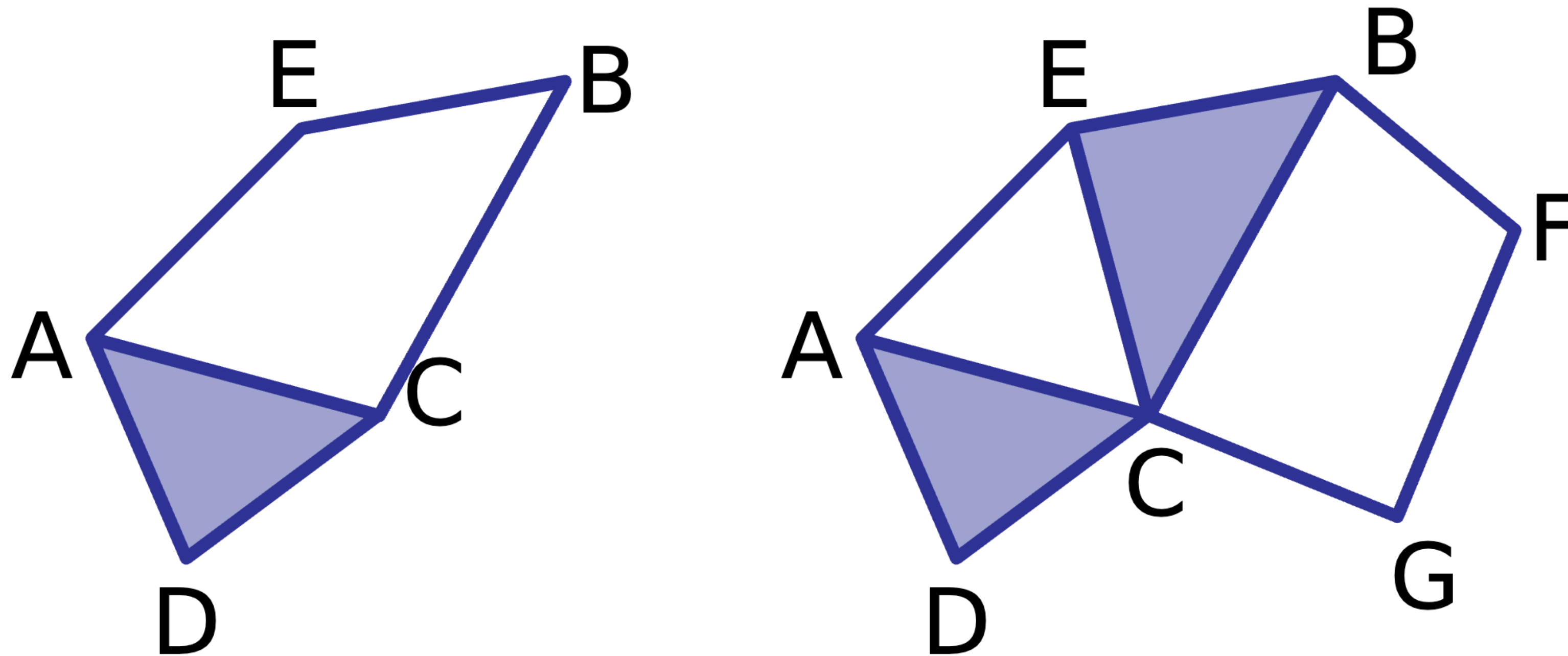
Write the **0**th and **1**st Betti curves for this filtration:



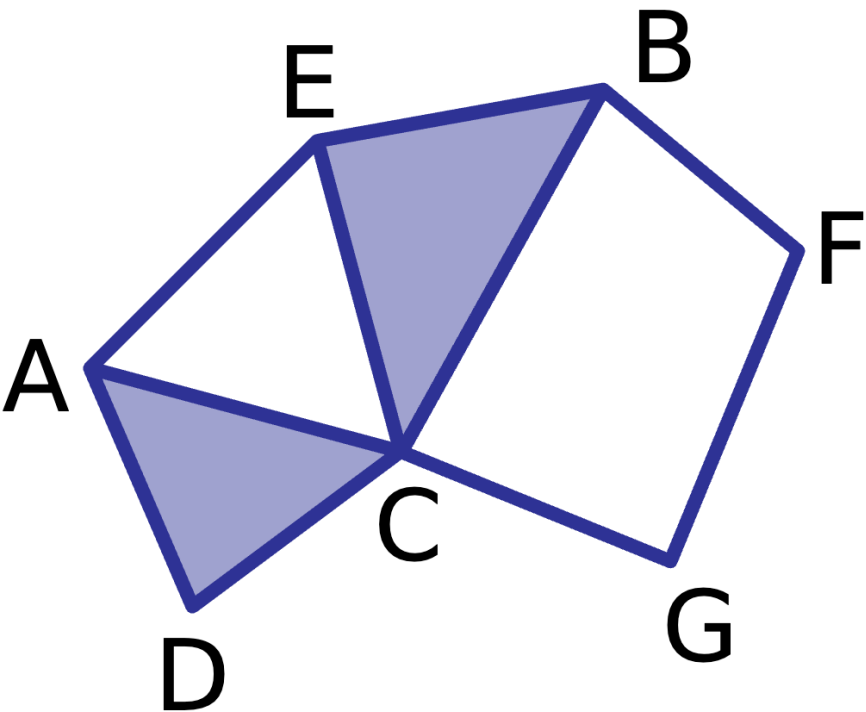
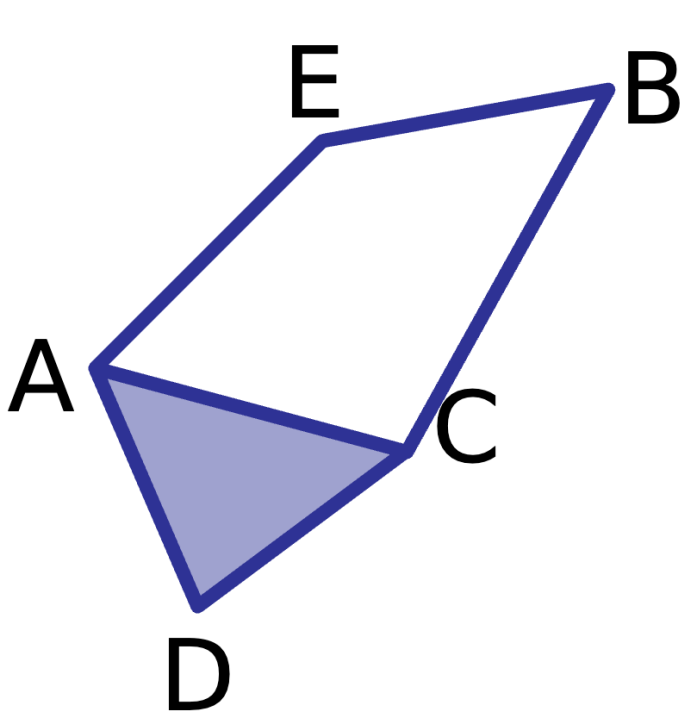
Induced Maps

Simplicial Maps

A simplicial map between abstract simplicial complexes $f: K \rightarrow L$ is induced by a map on the vertices $f: V(K) \rightarrow V(L)$.



Big Diagram of Vector Spaces



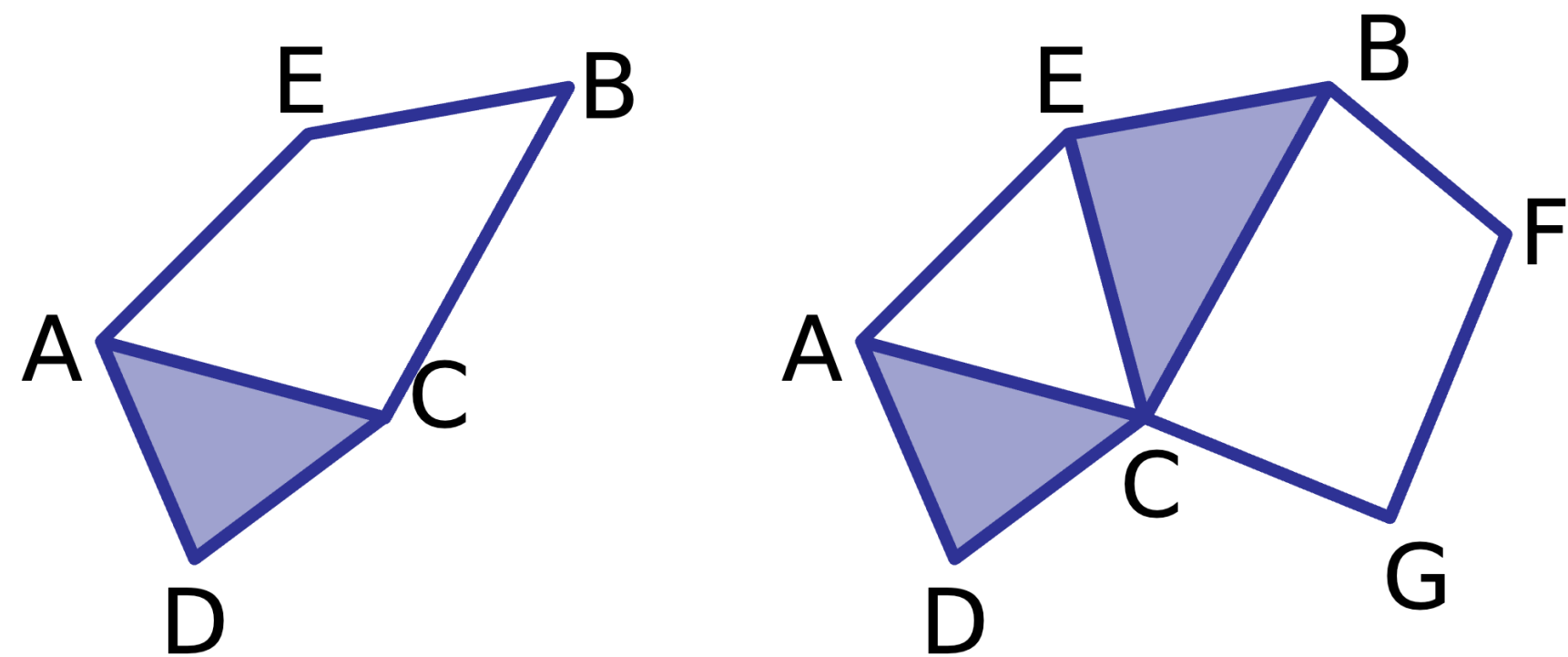
$$\cdots \longrightarrow C_2(K) \xrightarrow{\partial_2^K} C_1(K) \xrightarrow{\partial_1^K} C_0(K) \xrightarrow{\partial_0^K} 0$$

$$\cdots \longrightarrow C_2(L) \xrightarrow{\partial_2^L} C_1(L) \xrightarrow{\partial_1^L} C_0(L) \xrightarrow{\partial_0^L} 0$$

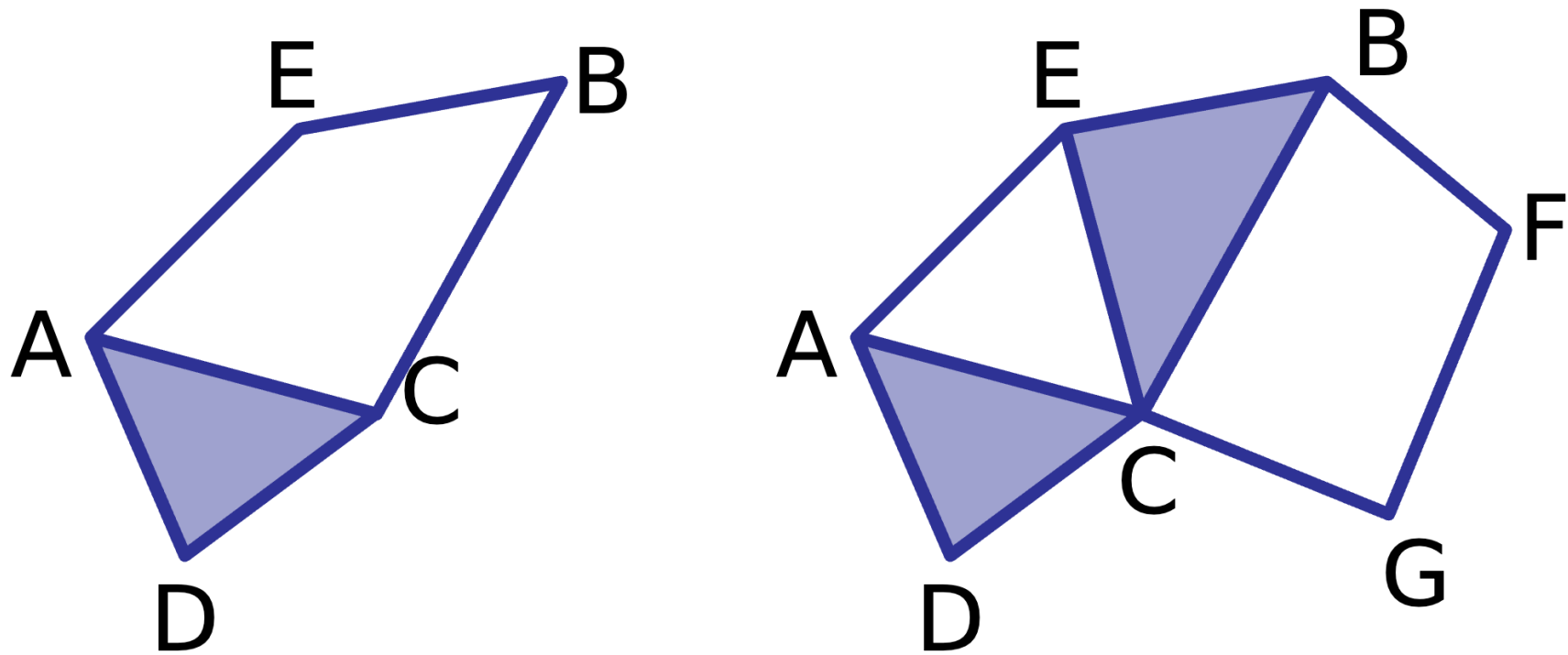
Induced Map

Given a simplicial map $f: K \rightarrow L$, the induced map on homology is defined by

$$\begin{aligned} f_* : H_p(K) &\longrightarrow H_p(L) \\ [\alpha] &\longmapsto [f_\#(\alpha)] \end{aligned}$$



Commutative Diagram

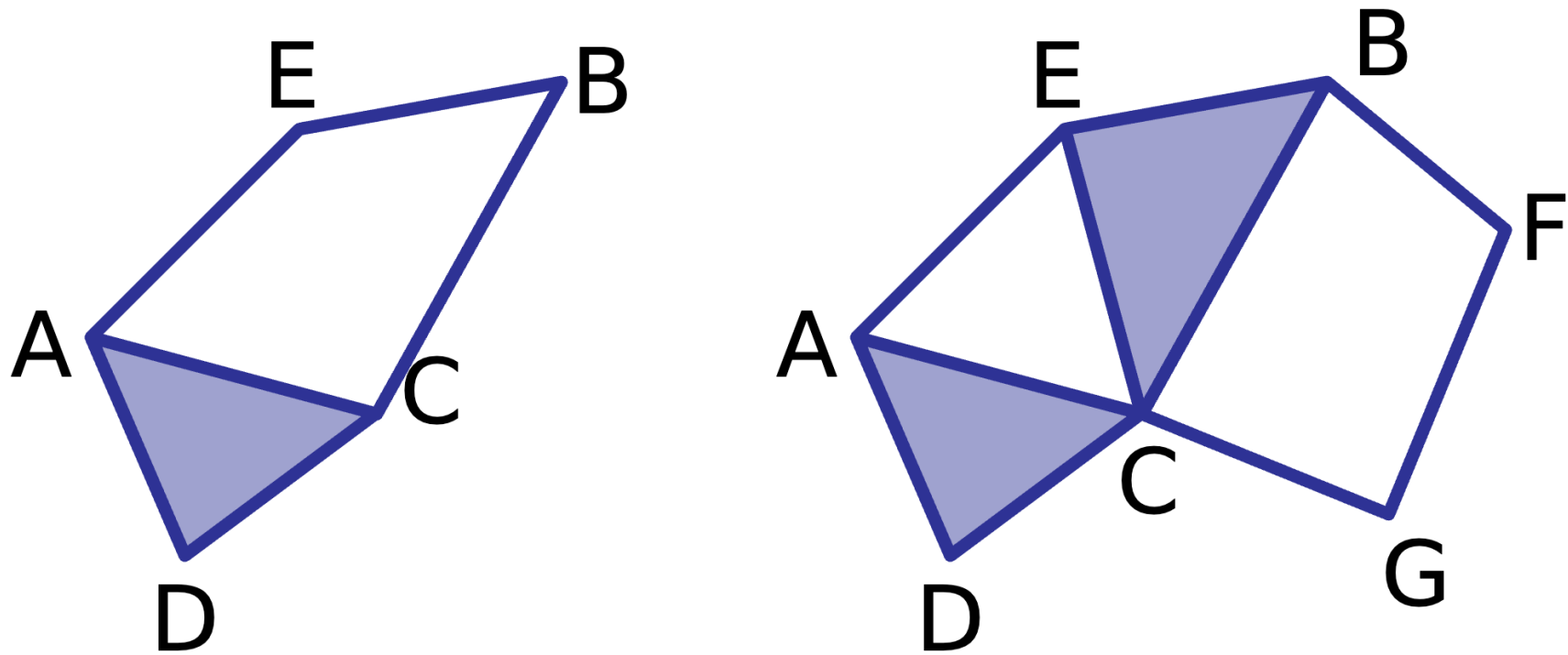


A diagram commutes if any path of functions is equal.

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & C_2(K) & \xrightarrow{\partial_2^K} & C_1(K) & \xrightarrow{\partial_1^K} & C_0(K) \xrightarrow{\partial_0^K} 0 \\
 & & \downarrow f_{\#} & & \downarrow f_{\#} & & \downarrow f_{\#} \\
 \cdots & \longrightarrow & C_2(L) & \xrightarrow{\partial_2^L} & C_1(L) & \xrightarrow{\partial_1^L} & C_0(L) \xrightarrow{\partial_0^L} 0
 \end{array}$$

Claim: $f_{\#} \circ \partial^K = \partial^L \circ f_{\#}.$

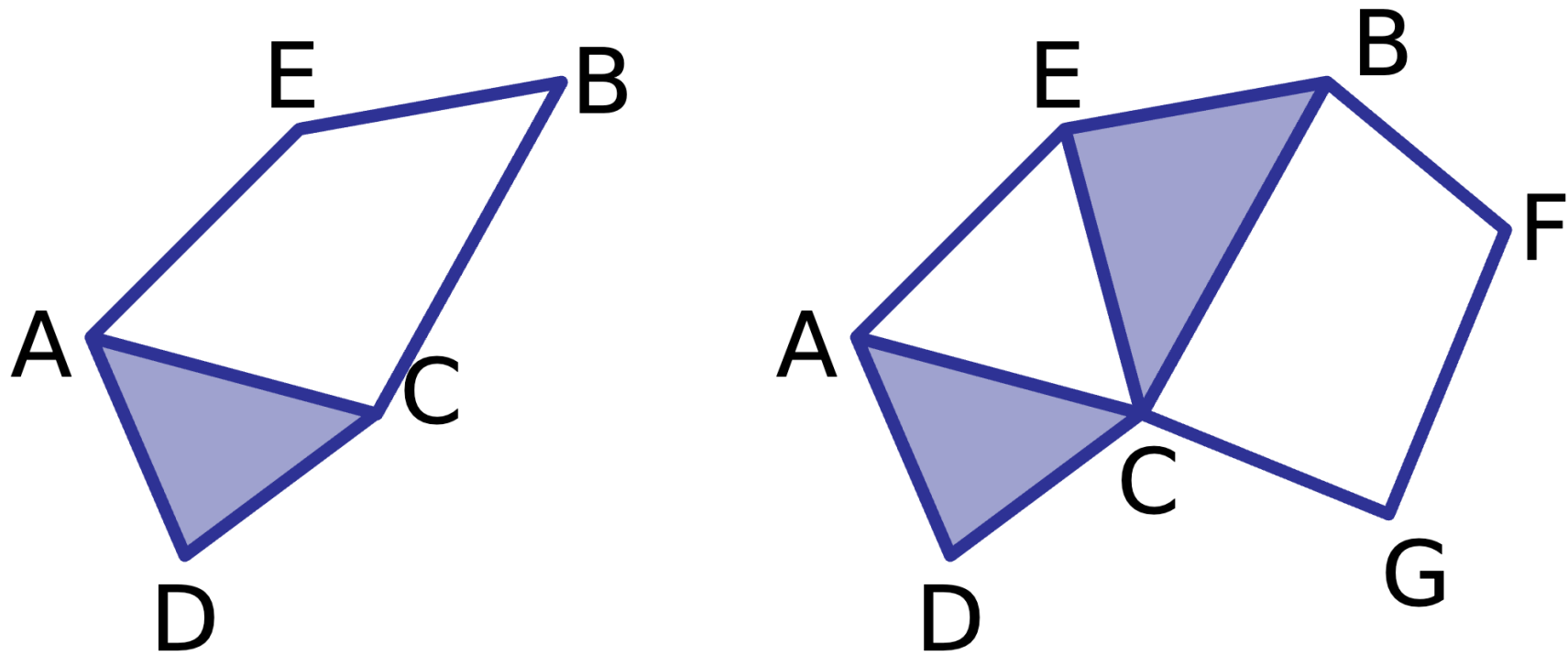
Effect on Cycles



Cycles go to cycles!

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & C_2(K) & \xrightarrow{\partial_2^K} & C_1(K) & \xrightarrow{\partial_1^K} & C_0(K) \xrightarrow{\partial_0^K} 0 \\
 & & \downarrow f_\# & & \downarrow f_\# & & \downarrow f_\# \\
 \cdots & \longrightarrow & C_2(L) & \xrightarrow{\partial_2^L} & C_1(L) & \xrightarrow{\partial_1^L} & C_0(L) \xrightarrow{\partial_0^L} 0
 \end{array}$$

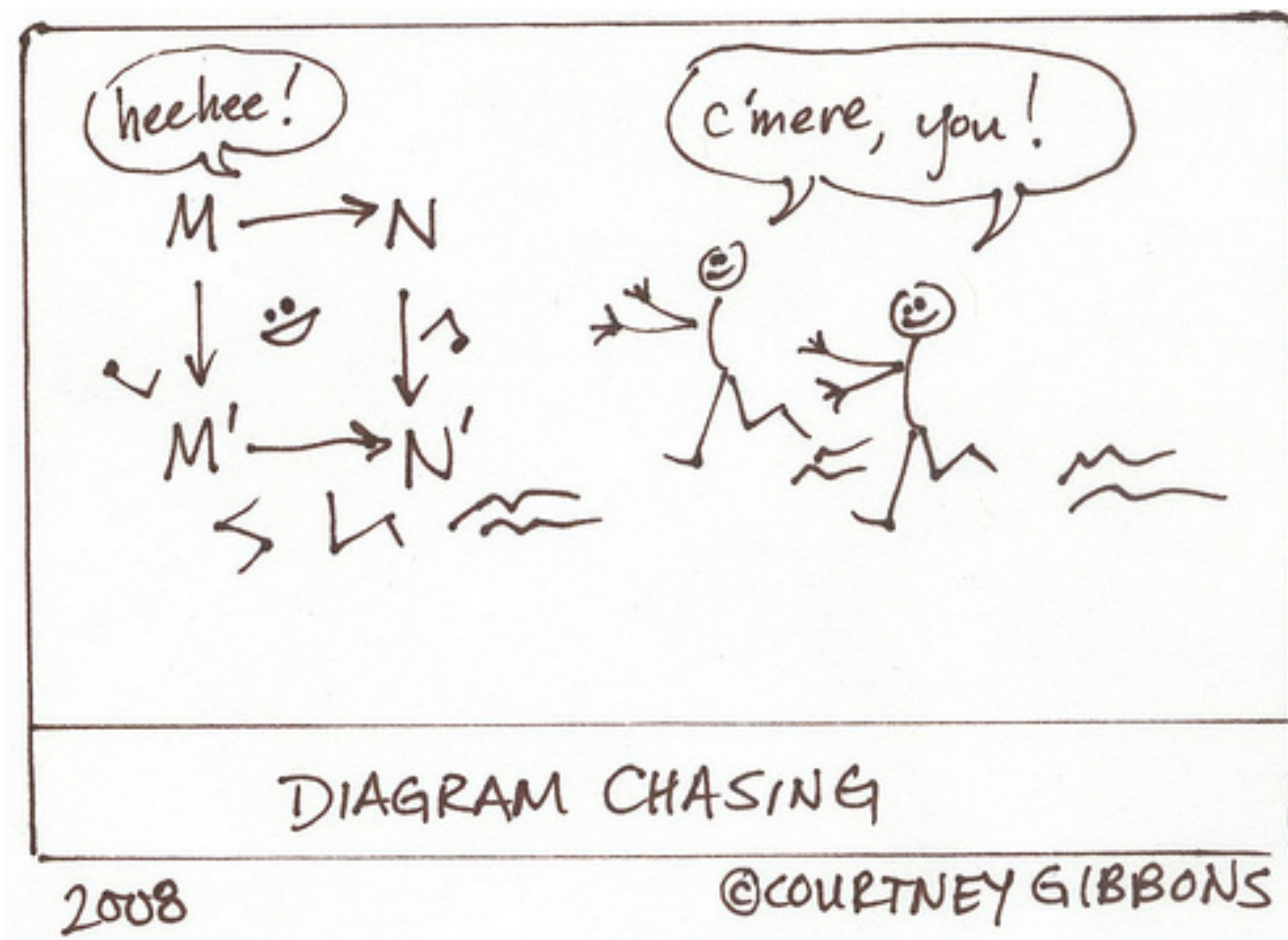
Effect on Boundaries



Boundaries go to boundaries!

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & C_2(K) & \xrightarrow{\partial_2^K} & C_1(K) & \xrightarrow{\partial_1^K} & C_0(K) \xrightarrow{\partial_0^K} 0 \\
 & & \downarrow f_\# & & \downarrow f_\# & & \downarrow f_\# \\
 \cdots & \longrightarrow & C_2(L) & \xrightarrow{\partial_2^L} & C_1(L) & \xrightarrow{\partial_1^L} & C_0(L) \xrightarrow{\partial_0^L} 0
 \end{array}$$

Diagram Chasing



Try it

If $H_1(K)$ generated by

$$[\alpha_1] = [AC + CD + AD]$$

$$[\alpha_2] = [BE + EL + CL + CB]$$

$$[\alpha_3] = [EF + FG + GL + EL]$$

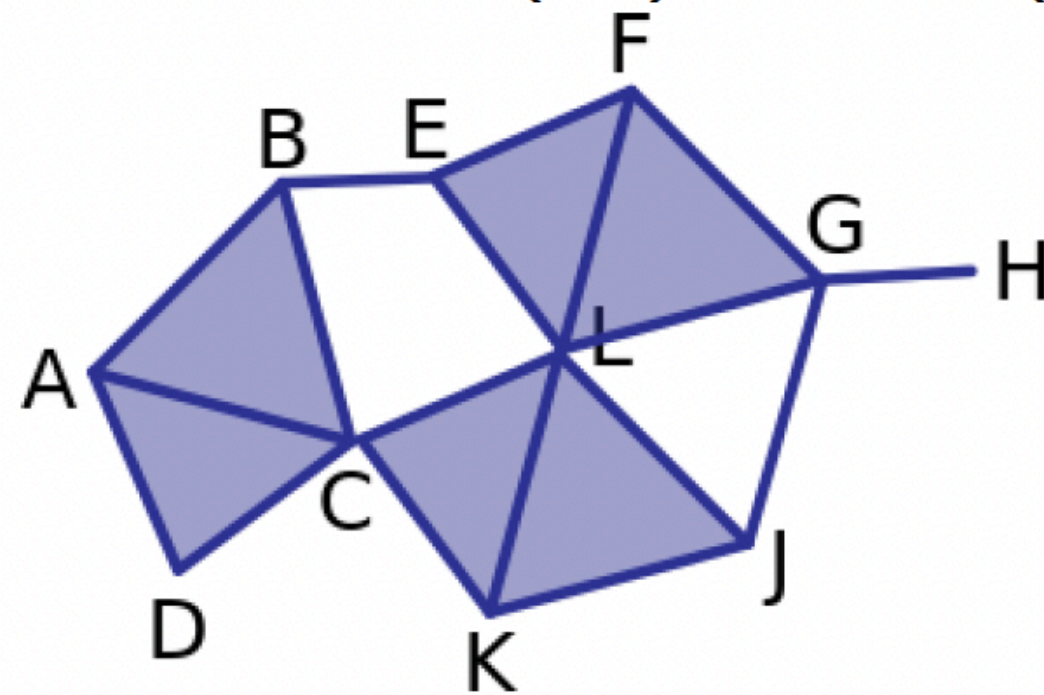
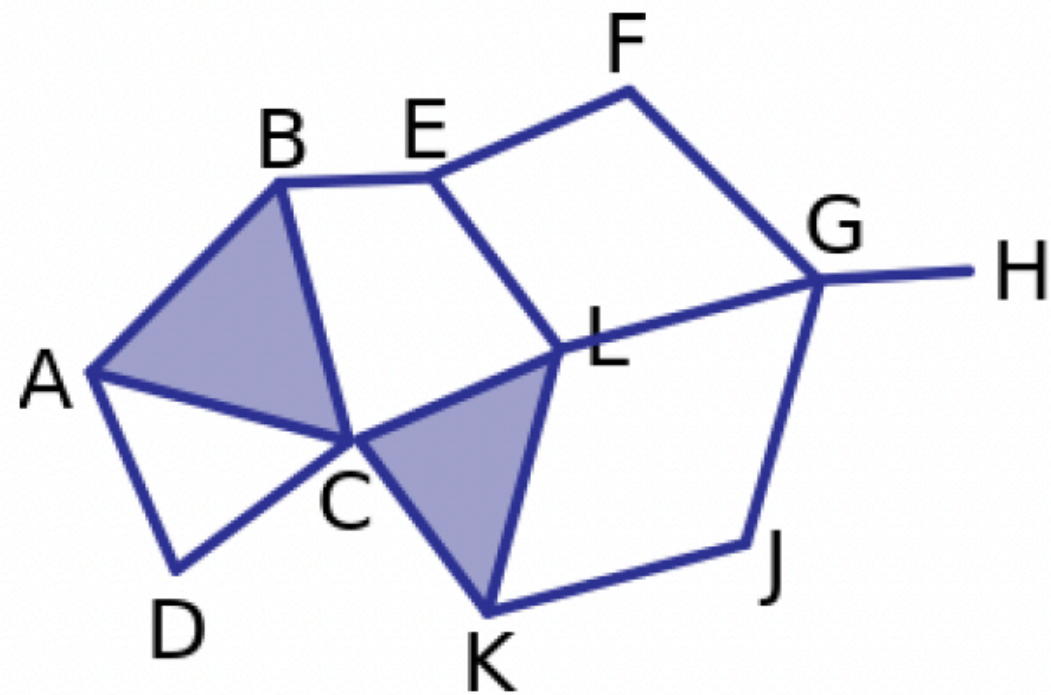
$$[\alpha_4] = [GL + GJ + JK + LK]$$

write the matrix representing $f_* : H_1(K) \rightarrow H_1(L)$ induced by inclusion.

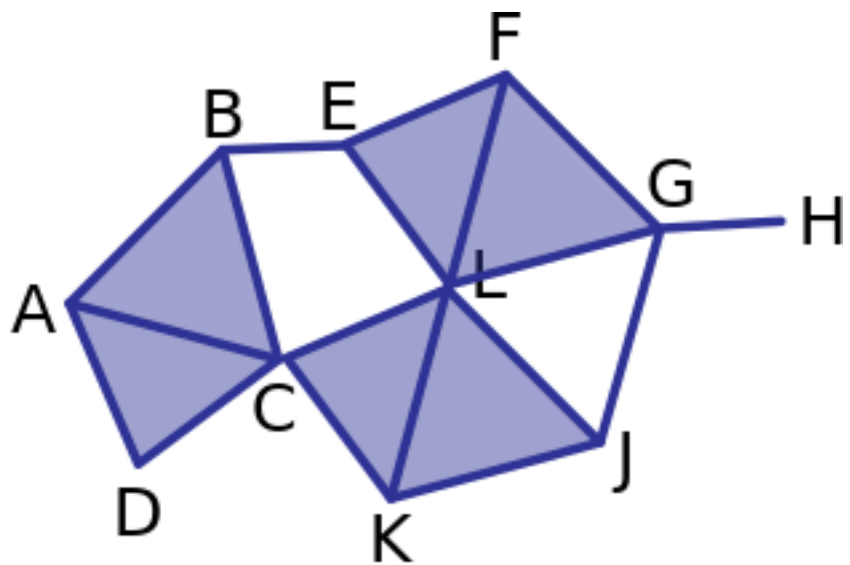
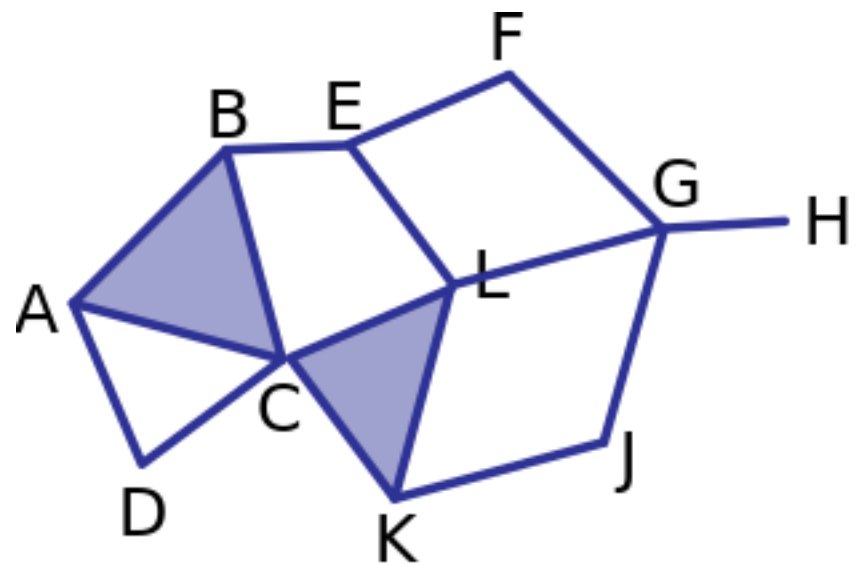
and $H_1(L)$ generated by

$$[\beta_1] = [BE + EL + CL + CB]$$

$$[\beta_2] = [GL + GJ + LJ]$$



Spare paper



$$\begin{array}{ccccccc} \dots & \longrightarrow & C_2(K) & \xrightarrow{\partial_2^K} & C_1(K) & \xrightarrow{\partial_1^K} & C_0(K) \xrightarrow{\partial_0^K} 0 \\ & & \downarrow f_\# & & \downarrow f_\# & & \downarrow f_\# \\ \dots & \longrightarrow & C_2(L) & \xrightarrow{\partial_2^L} & C_1(L) & \xrightarrow{\partial_1^L} & C_0(L) \xrightarrow{\partial_0^L} 0 \end{array}$$

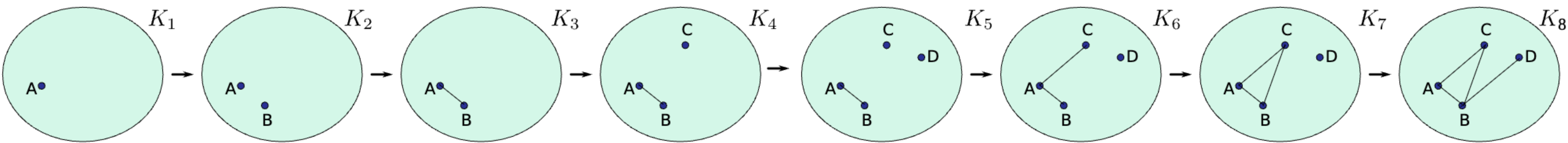
Persistence Module

Given a filtration, the p -dimensional persistence module is

$$H_p(\mathcal{F}) : 0 = H_p(K_0) \rightarrow H_p(K_1) \rightarrow H_p(K_2) \rightarrow \cdots \rightarrow H_p(K_n) = H_p(K)$$

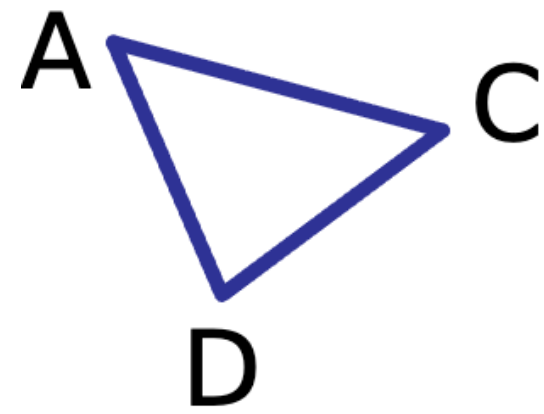
with maps $h_p^{i,j} : H_p(K_i) \rightarrow H_p(K_j)$ induced by inclusion

What's going on algebraically?

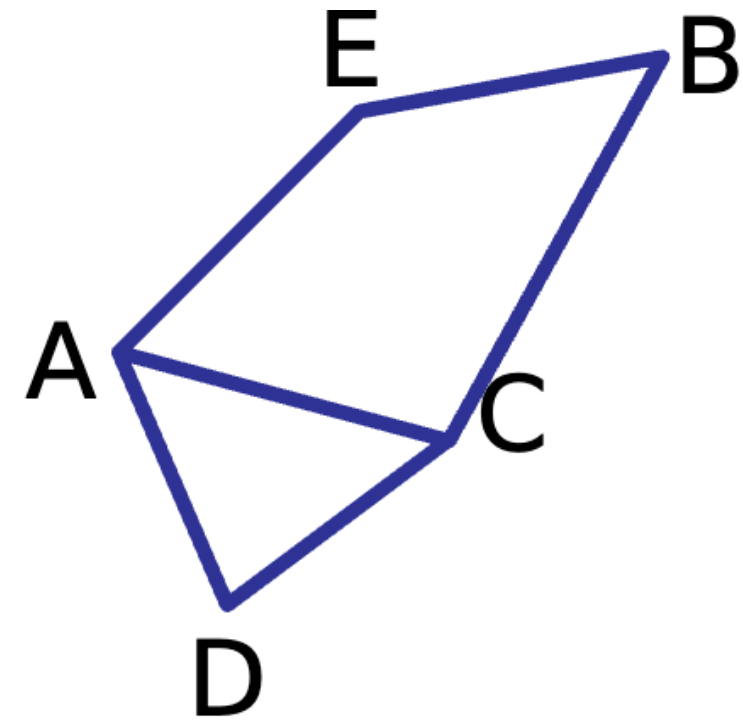


$$H_0(K_1) \longrightarrow H_0(K_2) \longrightarrow H_0(K_3) \longrightarrow H_0(K_4) \longrightarrow H_0(K_5) \longrightarrow H_0(K_6) \longrightarrow H_0(K_7) \longrightarrow H_0(K_8)$$

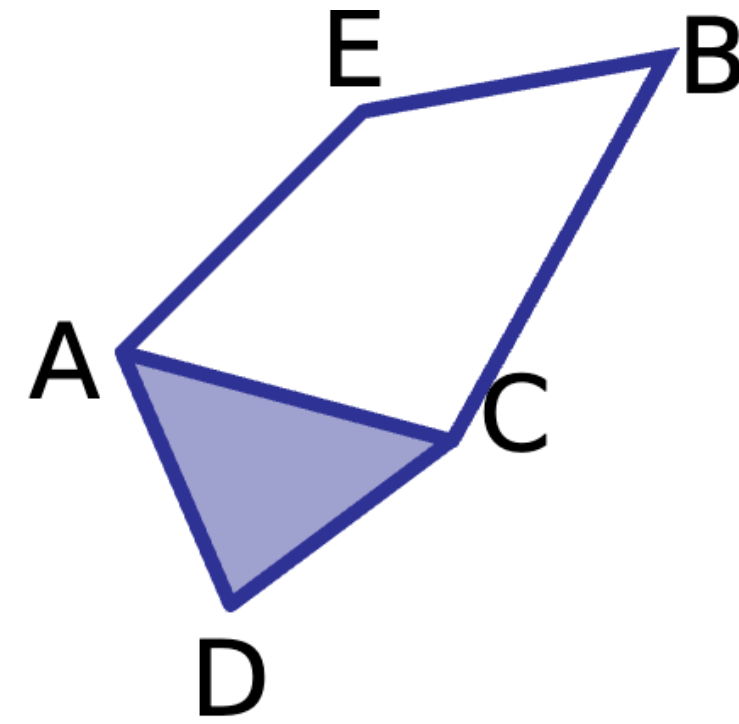
Another Example



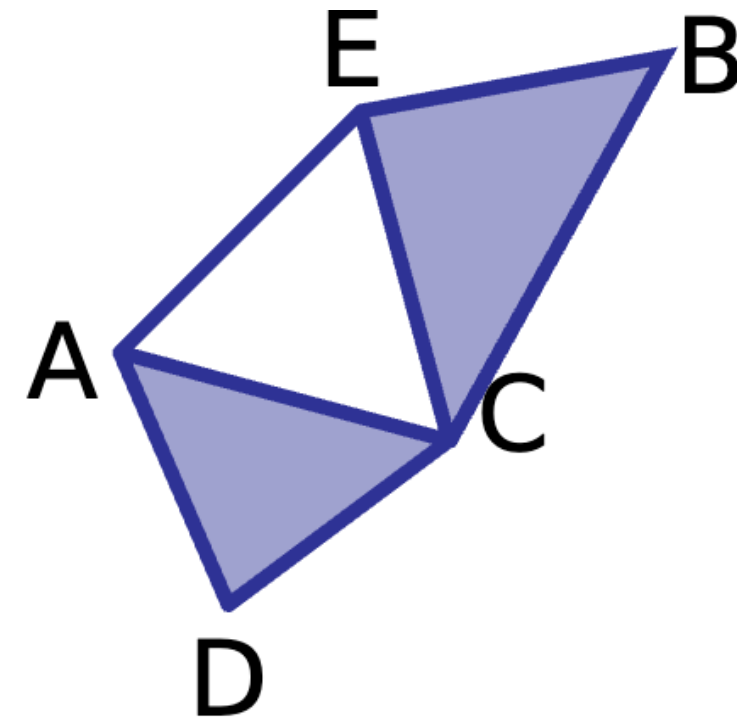
K_0



K_1



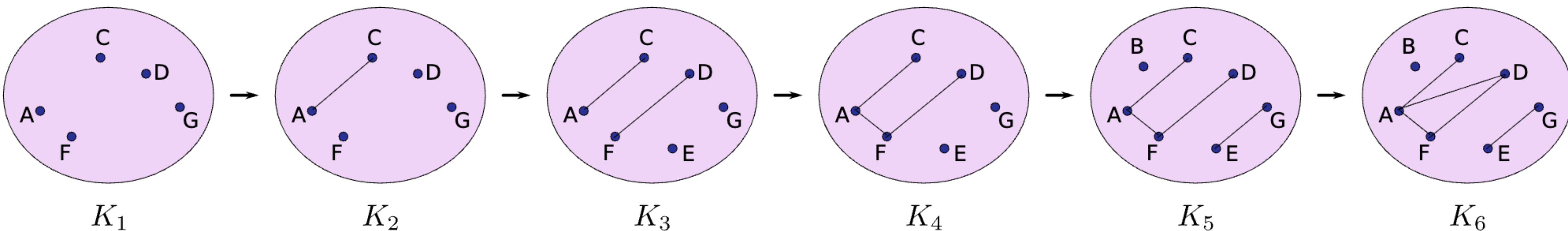
K_2



K_3

$$H_1(K_0) \xrightarrow{f_*} H_1(K_1) \xrightarrow{g_*} H_1(K_2) \xrightarrow{h_*} H_1(K_2)$$

Try it: Where do the generators go?



$$H_0(K_1) \longrightarrow H_0(K_2) \longrightarrow H_0(K_3) \longrightarrow H_0(K_4) \longrightarrow H_0(K_5) \longrightarrow H_0(K_6)$$

Next time: Persistent Homology!