Persistence

Project

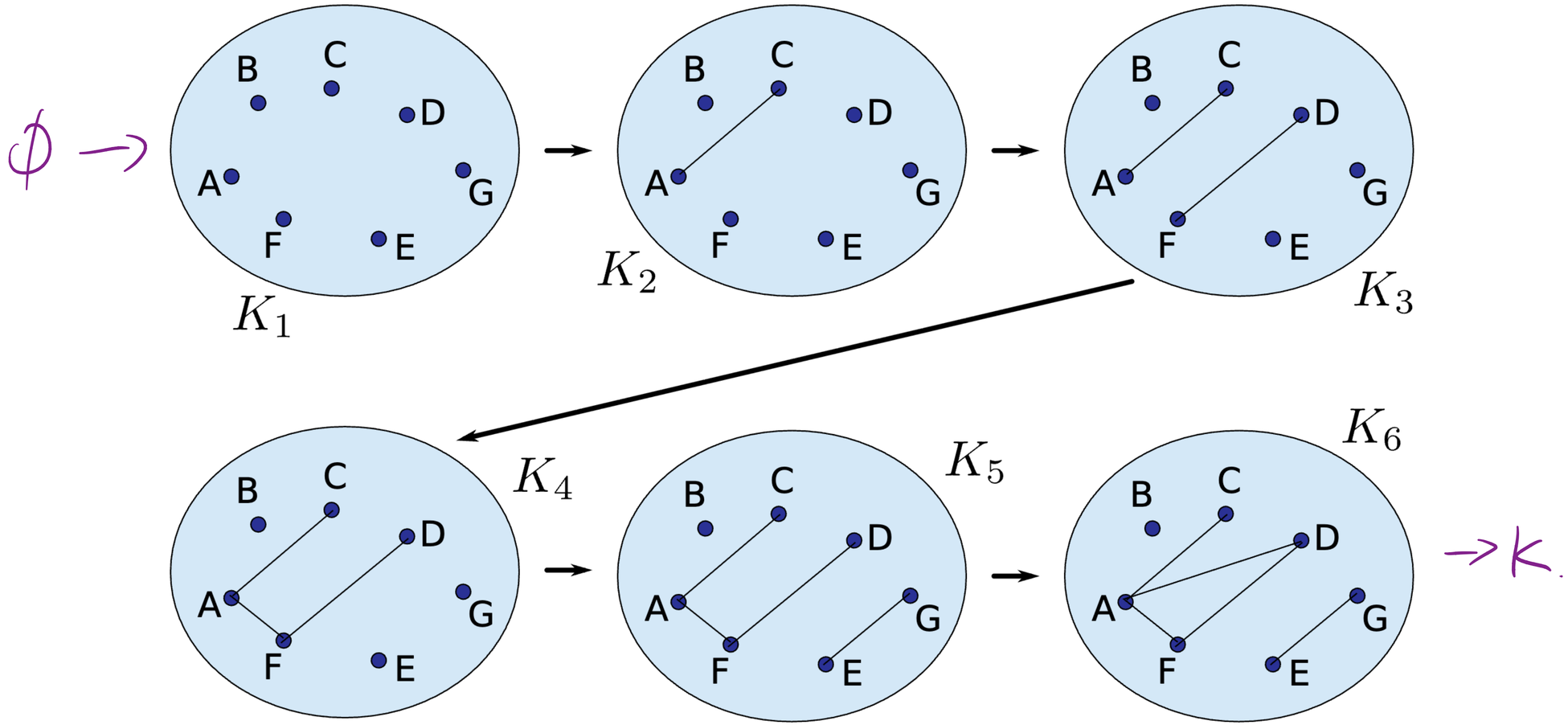
HW 2 Due 22/10 midnight.

- HW3 (paper chase) posted, due Nov 1st; read the instructions **carefully**
- Paper list also posted → 3 pg
- Schedule a meeting with me to discuss the project proposal between Nov 5 - 6
- Written proposal due Nov 11 → 3 pg
- Feedback on proposal Nov 18
- Dec 15 slides due
- Dec 16 presentations
- Final report? Due Jan 25 → 10-15 pg

Exam Feb 12.

Simplicial Complex Filtration

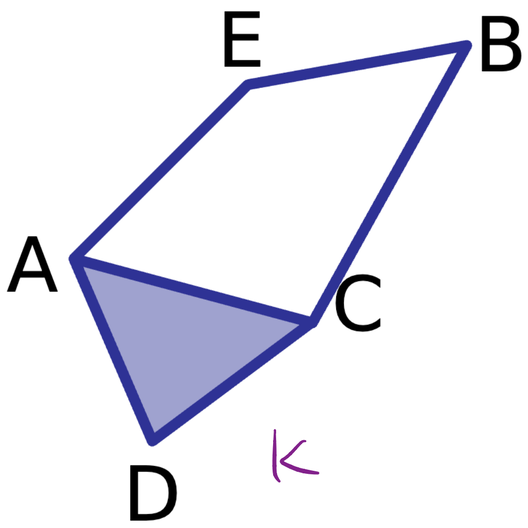
A *filtration* $\mathcal{F} = \mathcal{F}(K)$ of a simplicial complex K is a nested sequence of its subcomplexes $\mathcal{F} : \emptyset = K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n = K.$

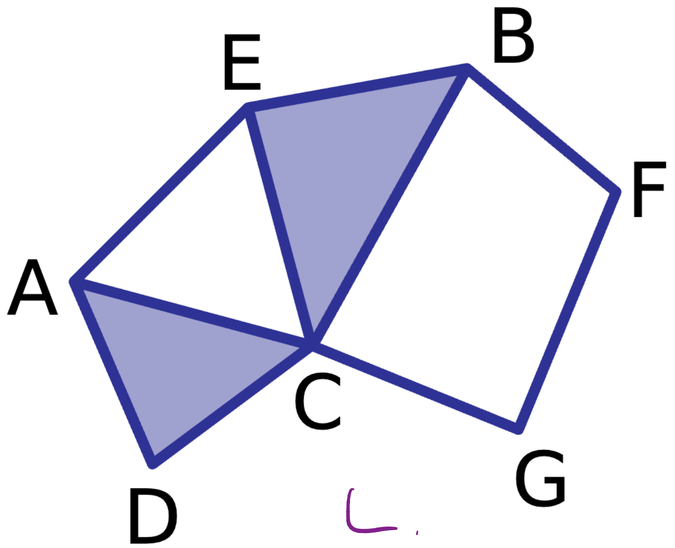


Simplicial Maps

A *simplicial map* between abstract simplicial complexes $f : K \to L$ is induced by a map on the vertices $f : V(K) \to V(L)$. For every $\sigma \in K, f(\sigma) \in L$.

Cheap trick: we can use inclusion!





$f(ADC) = ADC$ (ADC $\in K$, ADC $\in L$)

$f(AE+EB+BC+AC) = AE+EB+BC+AC$

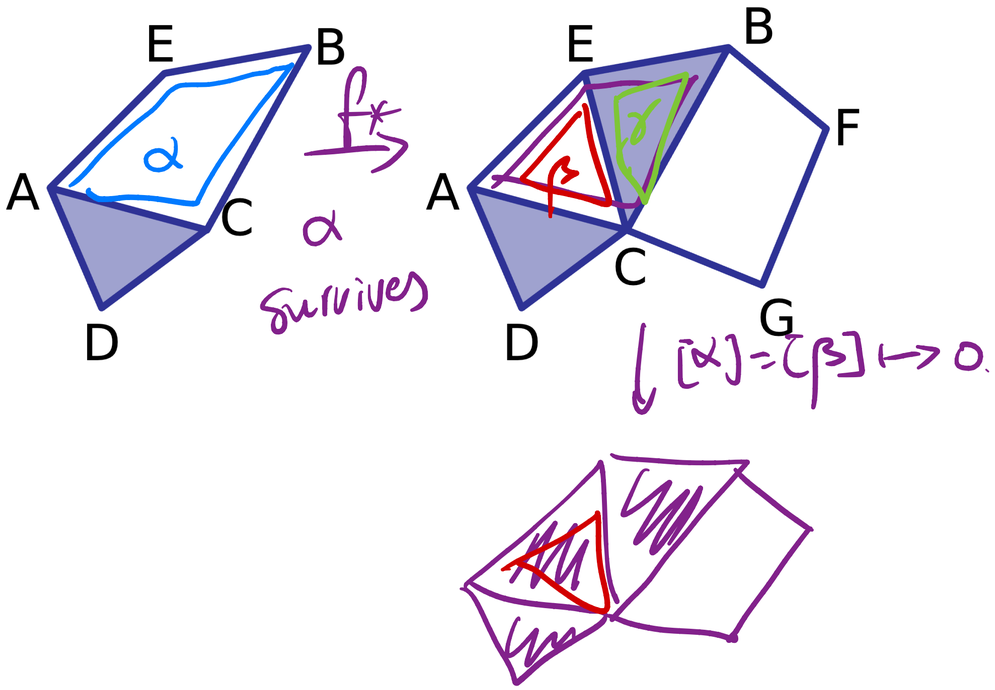
Induced Map

Given a simplicial map $f : K \to L$, the induced map on homology is defined by

$$f_* : H_p(K) \longrightarrow H_p(L)$$
$$[\alpha] \longmapsto [f_\#(\alpha)]$$

a hole

still a hole?



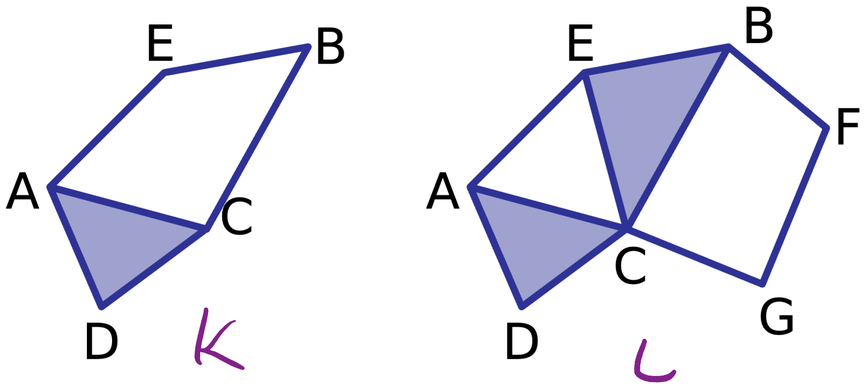
$\gamma = \partial_2(EBC)$

$\alpha = \beta + \gamma$

$[\alpha] = [\beta]$

$f_*([\beta]) = 0$

Commutative Diagram



A diagram commutes if any path of functions is equal.

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & C_2(K) & \xrightarrow{\partial_2^K} & C_1(K) & \xrightarrow{\partial_1^K} & C_0(K) & \xrightarrow{\partial_0^K} & 0 \\ & & \downarrow f_\# & & \downarrow f_\# & & \downarrow f_\# & & \\ \cdots & \longrightarrow & C_2(L) & \xrightarrow{\partial_2^L} & C_1(L) & \xrightarrow{\partial_1^L} & C_0(L) & \xrightarrow{\partial_0^L} & 0 \end{array}$$

AC

$f(A) = C$
$f(C) = C$
$\Rightarrow f(AC) = C$

$f_\#(AC) = 0 - 0$

Claim: $f_\# \circ \partial^K = \partial^L \circ f_\#$.

[cycles go to cycles
boundaries go to boundaries.

Example

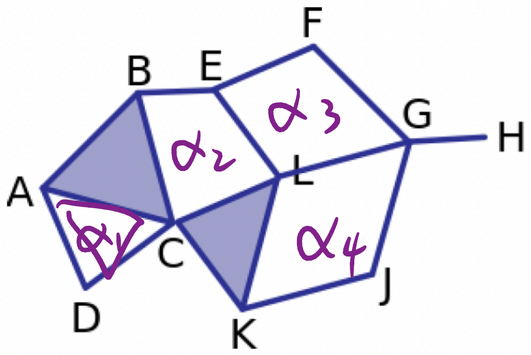
If $H_1(K)$ generated by

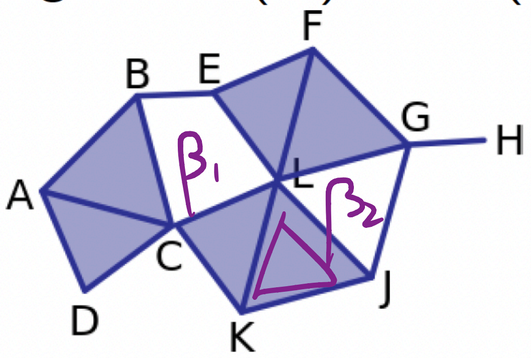
$$[\alpha_1] = [AC + CD + AD]$$
$$[\alpha_2] = [BE + EL + CL + CB]$$
$$[\alpha_3] = [EF + FG + GL + EL]$$
$$[\alpha_4] = [GL + GJ + JK + LK]$$

and $H_1(L)$ generated by

$$[\beta_1] = [BE + EL + CL + CB]$$
$$[\beta_2] = [GL + GJ + LJ]$$

write the matrix representing $f_* : H_1(K) \to H_1(L)$ induced by inclusion.





	α_1	α_2	α_3	α_4
β_1	0	1	0	0
β_2	0	0	0	1

Try it

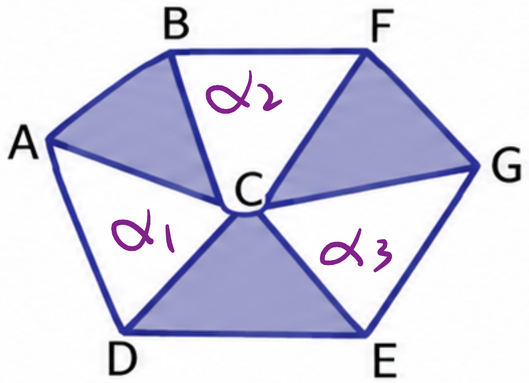
If $H_1(K)$ is generated by

$$[\alpha_1] = [AB + BC + CA]$$
$$[\alpha_2] = [CD + DE + EC]$$
$$[\alpha_3] = [BF + FG + GB]$$

and $H_1(L)$ is generated by

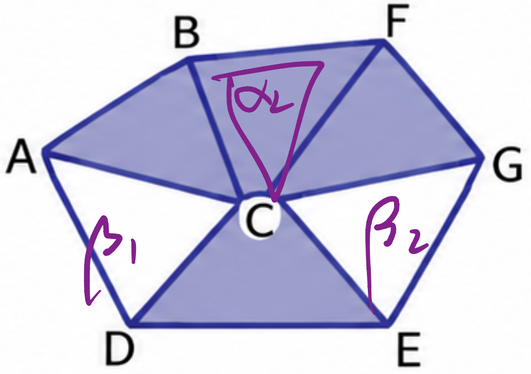
$$[\beta_1] = [AB + BF + FA]$$
$$[\beta_2] = [CD + DE + EC]$$

write the matrix representing $f_* : H_1(K) \to H_1(L)$ induced by inclusion.



K

$\hookrightarrow$



L

	α_1	α_2	α_3
β_1	1	0	0
β_2	0	0	1

Persistence Module

Given a filtration, the p-dimensional *persistence module* is

$$H_p(\mathcal{F}) : 0 = H_p(K_0) \to H_p(K_1) \to H_p(K_2) \to \cdots \to H_p(K_n) = H_p(K)$$

with maps $h_p^{i,j} : H_p(K_i) \to H_p(K_j)$ induced by inclusion

dim

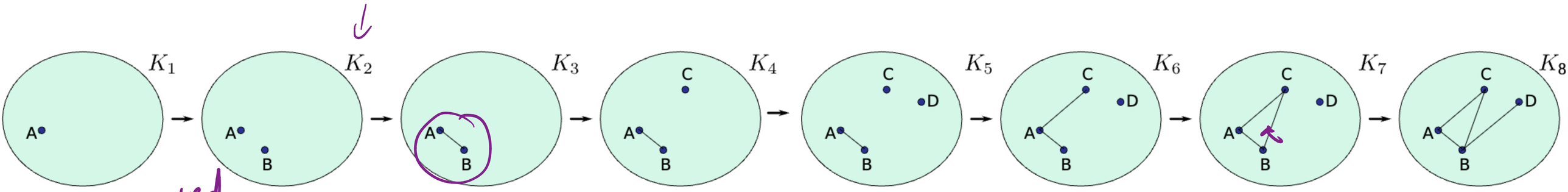
classes that existed here

survives?

$\alpha \in H_p(K_i)$ $\beta \in H_p(K_j)$.

$\alpha \hookrightarrow \beta$, $\beta \neq 0$.

What's going on algebraically?



connected components.

$$H_0(K_1) \longrightarrow H_0(K_2) \longrightarrow H_0(K_3) \longrightarrow H_0(K_4) \longrightarrow H_0(K_5) \longrightarrow H_0(K_6) \longrightarrow H_0(K_7) \longrightarrow H_0(K_8)$$

$[A] \longmapsto [A] \longmapsto [A] \longmapsto [A] \longmapsto [A] \longmapsto [A] \longmapsto [A]$

$[B] \longmapsto [B]$

$[C] \longmapsto [C]$

$[D] \longmapsto [D] \longmapsto [D]$

$[A]+[B] \longmapsto [A]+[A]$

$=0$

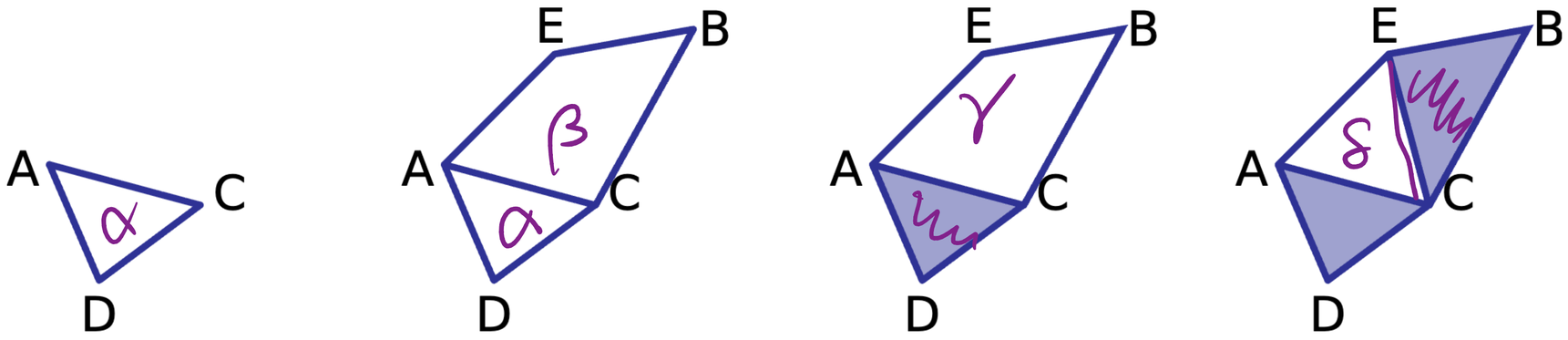
$\{[A], [B]\}$

$\{[A], [A]+[B]\}$

$H_1(K_7)$

$[\ \ ]$

Another Example



K_0 K_1 K_2 K_3

$$H_1(K_0) \xrightarrow{h_1^{0,1}} H_1(K_1) \xrightarrow{h_1^{1,2}} H_1(K_2) \xrightarrow{h_1^{2,3}} H_1(K_3)$$

$[\alpha] \longmapsto [\alpha] \longmapsto 0$

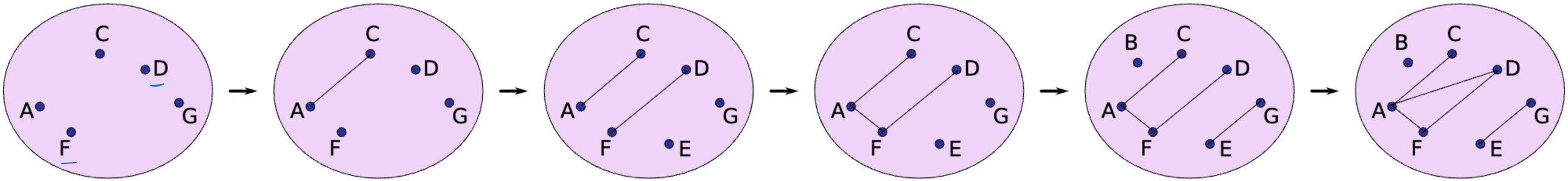
$[\beta] \longmapsto [\gamma] \longmapsto [\delta]$.

$\delta = \gamma + \partial(BCE)$

$\Rightarrow [\delta] = [\gamma]$.

elder rule: the feature that is born later dies.

Try it: Where do the generators go?



K_1 K_2 K_3 K_4 K_5 K_6

$$H_0(K_1) \longrightarrow H_0(K_2) \longrightarrow H_0(K_3) \longrightarrow H_0(K_4) \longrightarrow H_0(K_5) \longrightarrow H_0(K_6)$$

[A] ⟼ [A] ⟼ [A] ⟼ [A] ⟶ [A]

[C] ⟼ [C] ⟼ [B]

[D] ⟼ [D] ⟼ [D]

[F] ⟼ [F] ⟼ [E] ⟼ [E] ⟶ [E]

[G] ⟼ [G] ⟶ [G] ⟼ [G]

[D] + [F] ⟼ [D] + [F] ⟼ 0

The pth Persistent Homology Group

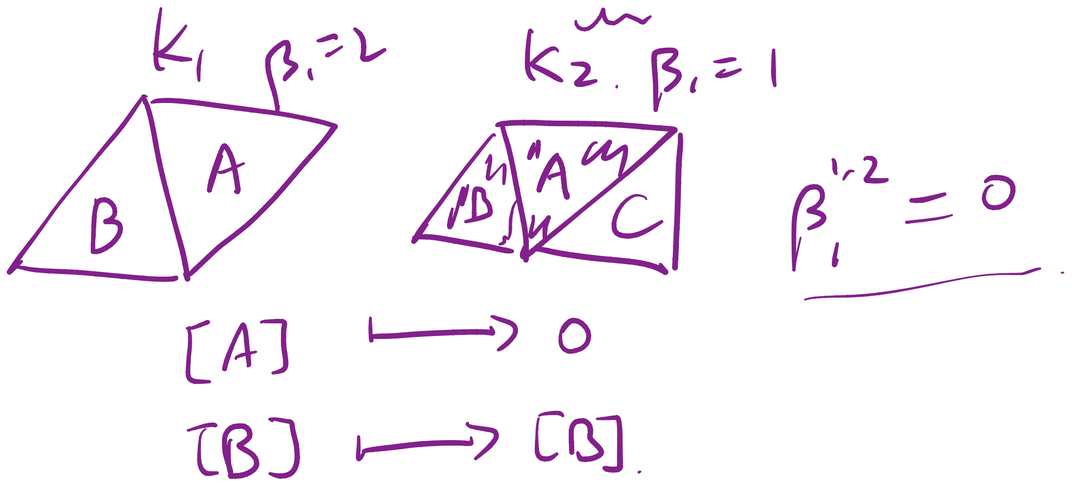
$$H_p(K_0) \to H_p(K_1) \xrightarrow{h_p^{1,2}} H_p(K_2) \to \cdots \to H_p(K_i) \to \cdots \to H_p(K_j) \to \cdots \to H_p(K_n)$$

The *pth persistent homology groups* are the images induced by inclusion

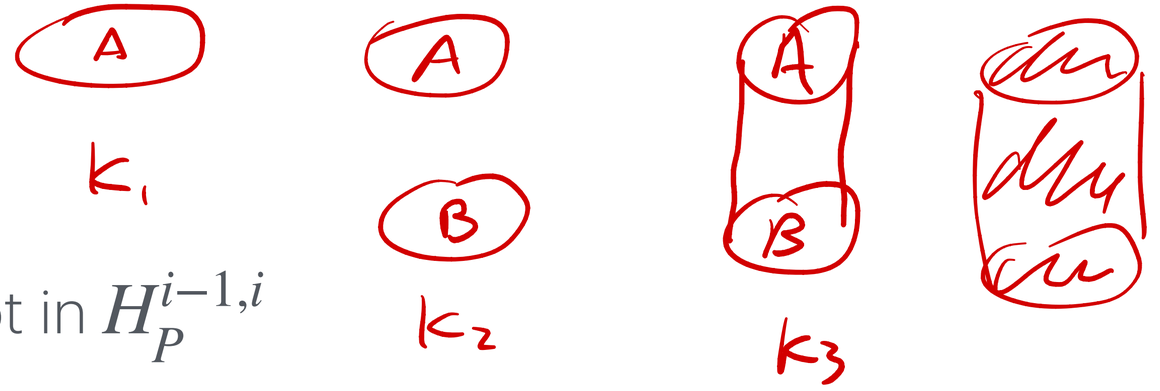
$$H_p^{i,j} = \operatorname{Im}\left(H_p(K_i) \to H_p(K_j)\right).$$

The pth persistent Betti numbers are the ranks

$$\beta_p^{i,j} = \operatorname{rank} H_p^{i,j}$$

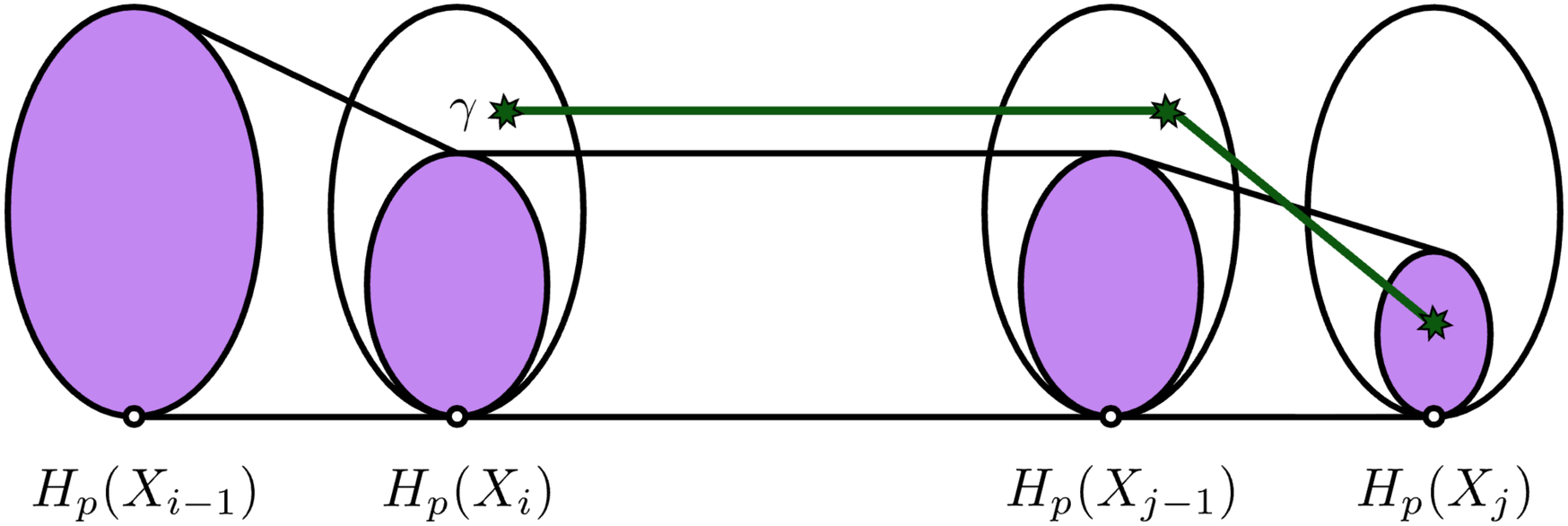


Birth and Death

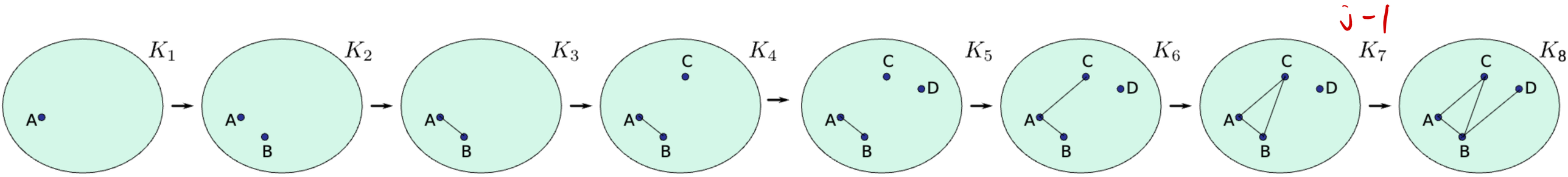


Birth: A class $\gamma \in H_p(K_i)$ is born at K_i if it is not in $H_P^{i-1,i}$

Death: The class γ dies entering K_j if merges with an older class. Specifically, if $h_p^{i,\,j-1}(\gamma) \notin H_p^{i-1,\,j-1}$ but $h_p^{i,\,j}(\gamma) \in H_p^{i-1,\,j}$.



When are generators born and die?



$$H_0(K_1) \longrightarrow H_0(K_2) \longrightarrow H_0(K_3) \longrightarrow H_0(K_4) \longrightarrow H_0(K_5) \longrightarrow H_0(K_6) \longrightarrow H_0(K_7) \longrightarrow H_0(K_8)$$

[A] ⟼ [A] ⟼ [A] ⟼ [A] ⟼ [A] ⟼ [A] ⟼ [A] ⟼ [A]

[B] ⟼ [B]

[C] ⟼ [C]

[D] ⟼ [D] ⟼ [D]

i = 2

j = 3

i = 4

j = 6

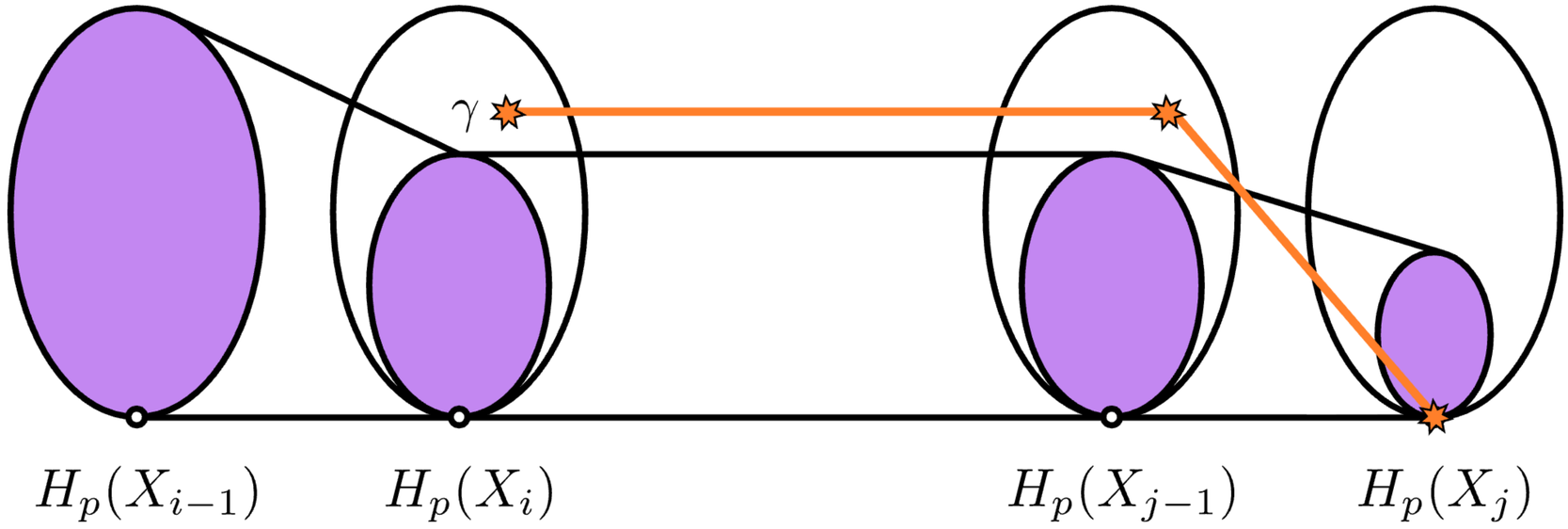
i = 5

j = 8

Birth and Death (Book Version)

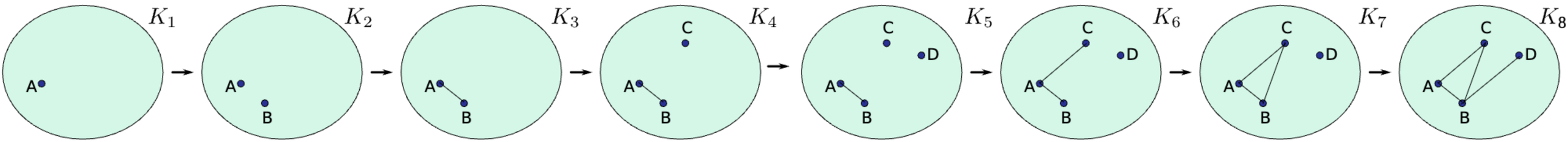
Birth: A class $\gamma \in H_p(K_i)$ is born at K_i if it is not in $H_P^{i-1,i}$

Death: The class γ dies entering K_j if $\gamma \in H_p(X_{j-1})$ is not trivial, but $h_p^{i-1,\,j}(\gamma) = 0$.



Now where do the elements go?

many but not all.



$$H_0(K_1) \longrightarrow H_0(K_2) \longrightarrow H_0(K_3) \longrightarrow H_0(K_4) \longrightarrow H_0(K_5) \longrightarrow H_0(K_6) \longrightarrow H_0(K_7) \longrightarrow H_0(K_8)$$

$0 \longmapsto 0 \longmapsto 0 \longmapsto 0 \longmapsto 0 \longmapsto 0$

$[A] \longmapsto [A] \longmapsto [A] \longmapsto [A] \longmapsto [A] \longmapsto [A]$

$[B] \longmapsto [C] \longmapsto [C] \longmapsto [D]$

$[A]+[B] \longmapsto 0 \qquad [A]+[C] \longmapsto [A]+[C] \longmapsto [A]+[D]$

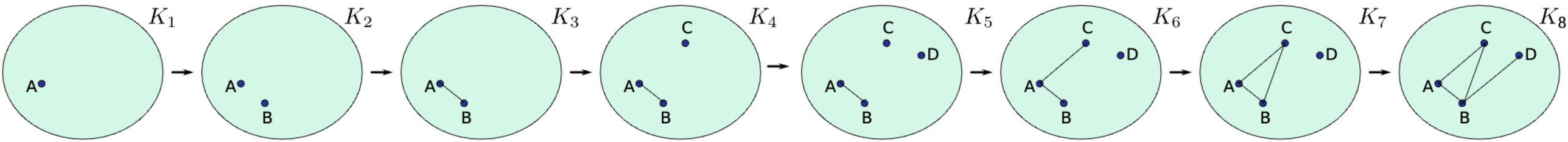
$[A]+[D] \longmapsto [A]+[D]$

$[C]+[D]$

$[A]+[C]+[D]$

Choosing a different basis

$\beta_0^{4,5} = 2$ $\beta_0^{4,6} = 1$



$$H_0(K_1) \longrightarrow H_0(K_2) \longrightarrow H_0(K_3) \longrightarrow H_0(K_4) \longrightarrow H_0(K_5) \longrightarrow H_0(K_6) \longrightarrow H_0(K_7) \longrightarrow H_0(K_8)$$

$[A] \longmapsto [A]$ $[A]$ $[A]$ $[A] \longrightarrow [A]$

$[A]+[B] \longmapsto 0$

$[A]+[C] \longmapsto 0$

$[A]+[D] \longmapsto 0$

$[C] = [A]+[C] = ([A]+[D])$ $h_1^{5,6} = [D]$

$([D]+[C])$

$i=4$ $j=6$

$[C] = h_0^{1,5}([A]) + h_0^{4,5}([C])$

Birth and Death Pairing (Book Version)

Let $[c]$ be a pth homology class that dies (maps to 0) entering X_j. Then, it is born at X_i if and only if there exists a sequence $i_1 \leq i_2 \leq \cdots \leq i_k = i$ for some $k \geq 1$ so that

- There is a c_{i_ℓ} where $[c_{i_\ell}]$ is born at X_{i_ℓ} for every $\ell \in \{1, \ldots, k\}$.
- $[c] = h_p^{i_1, j-1}([c_{i_1}]) + \cdots + h_p^{i_k, j-1}([c_{i_k}])$
- $i_k = i$ is the smallest possible value among any sequences have the above two properties.

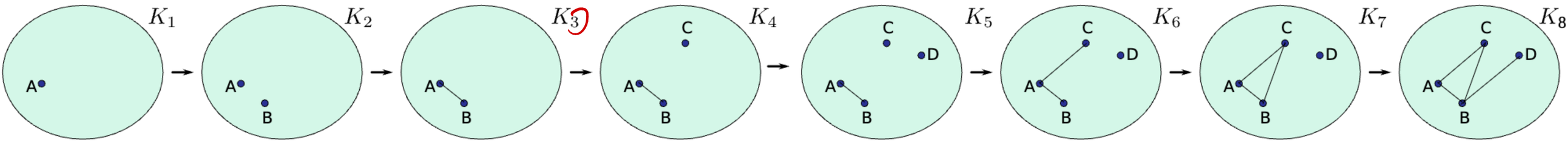
Counting Classes

$$0 \to H_p(K_1) \to H_p(K_2) \to H_p(K_3) \to \cdots \to H_p(K_n) \to 0$$

- Attach 0 vector space at the end → formal start & end point.
- Associate $n+1$ to $a_{n+1} = \infty$
- $\beta_p^{i,j} = \operatorname{rank} H_p^{i,j} = \operatorname{rank}\left(\operatorname{Im}\left(H_p(K_i) \to H_p(K_j)\right)\right)$ counts classes born **at or before** i and dying **after** j

Same example again

a=2 a=3 a=4 a=5 a=6 a=8



$$H_0(K_1) \longrightarrow H_0(K_2) \longrightarrow H_0(K_3) \longrightarrow H_0(K_4) \longrightarrow H_0(K_5) \longrightarrow H_0(K_6) \longrightarrow H_0(K_7) \longrightarrow H_0(K_8)$$

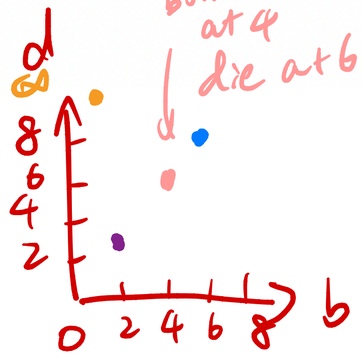
$$[A] \longmapsto [A] \longmapsto [A] \longmapsto [A] \longmapsto [A] \longmapsto [A] \longmapsto [A] \longmapsto [A]$$

$$[A+B] \longmapsto 0 \qquad [A+C] \longmapsto [A+C] \longmapsto 0$$

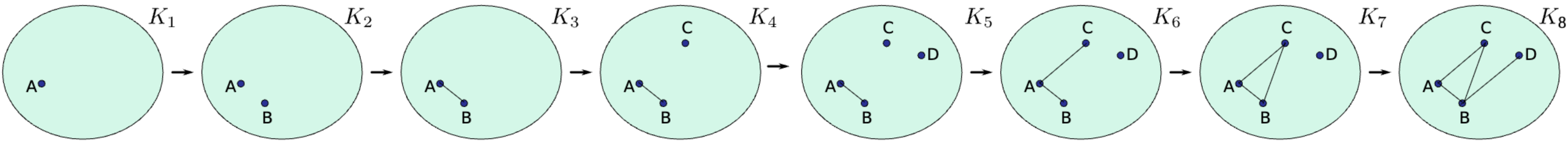
$3-2=1$ $\quad 6-4=2$

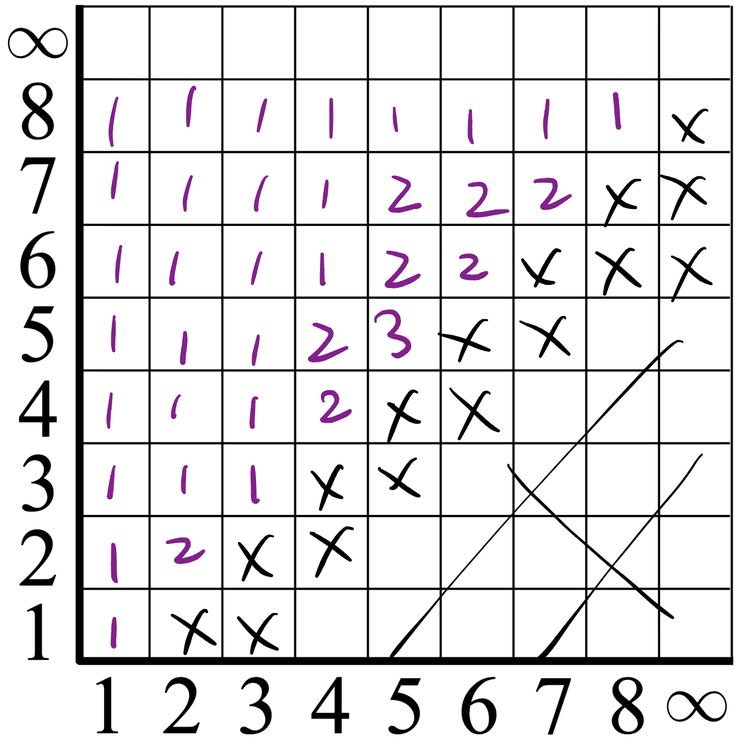
$$[A+D] \longmapsto [A+D] \longmapsto [A+D] \longmapsto 0$$

$8-5=3.$



Example





Determine $\beta_0^{i,\,j}$ for all pairs $i \leq j$

$\beta_0^{1,1}$

Persistence Pairing Function

How to get the number of classes born **at** i and dying **at** j ???

For $0 < i < j \leq n+1$, define

$$\mu_p^{i,j} = \begin{cases} (\beta_p^{i,j-1} - \beta_p^{i,j}) - (\beta_p^{i-1,j-1} - \beta_p^{i-1,j}), & i < j < n+1, \\ \beta_p^{i,j} - \beta_p^{i-1,j}, & i < j = n+1. \end{cases}$$

born $\leq i-1$
die $\geq j$

born $\leq i-1$
die $\geq j+1$

born $\leq i$
die $\geq j$

born $\leq i$
die $\geq j+1$

born $\leq i$
die $= j$

$-$

born $\leq i-1$
die $= j$

$=$

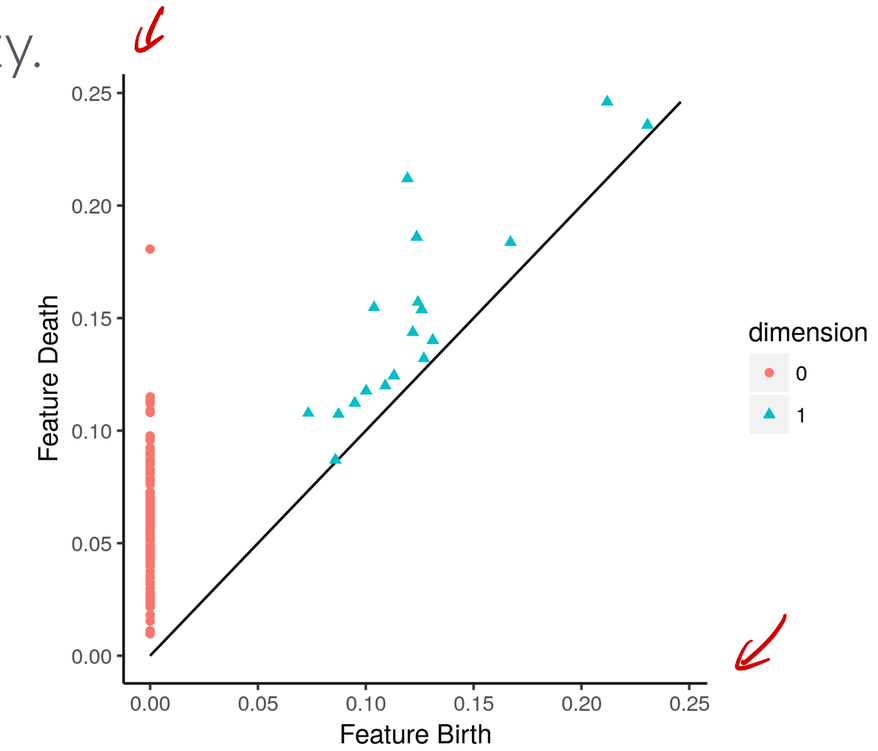
born $= i$
die $= j$

Persistence of a Class

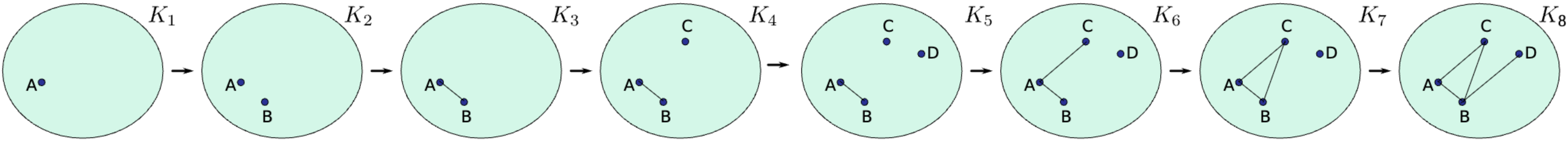
For $\mu_p^{i,\,j} \neq 0$, the persistence Pers$([c])$ of a class $[c]$ that is born at X_i and dies at X_j is defined as Pers$([c]) = a_j - a_i$. When $j = n+1$ with $a_{n+1} = \infty$, Pers$([c]) = \infty$.

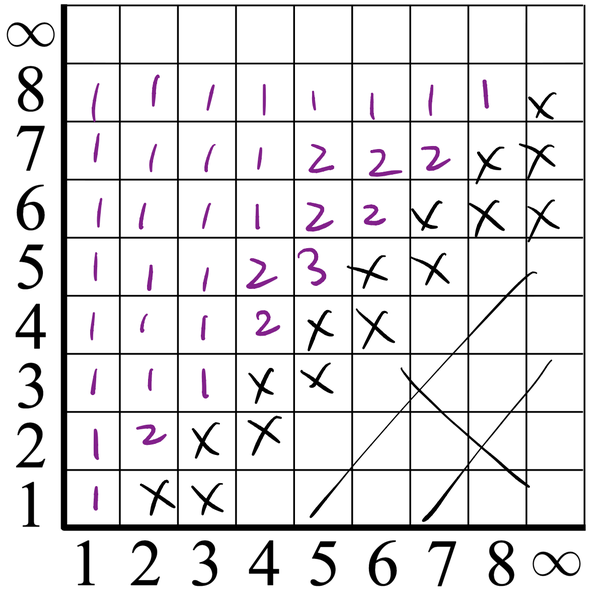
Persistence Diagram

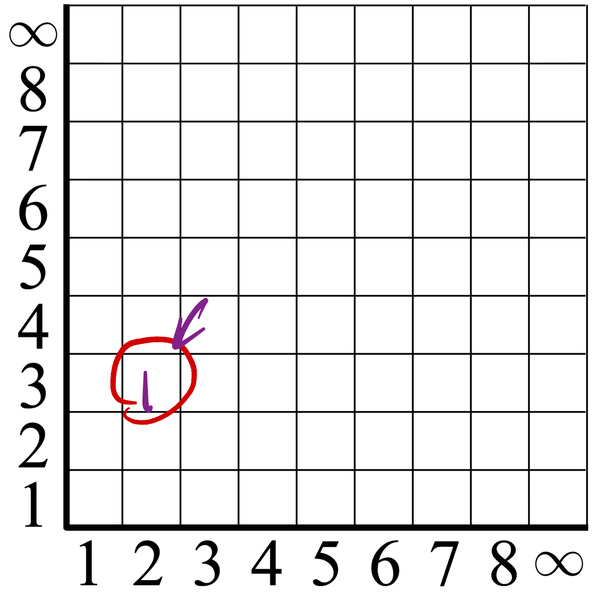
The persistence diagram $\mathrm{Dgm}_p(\mathcal{F})$ (also written $\mathrm{Dgm}_p(f)$) of a filtration induced by a function f is obtained by drawing a point $(a_i,\ a_j)$ with non-zero multiplicity $\mu_p^{i,\ j}(i < j)$, on the extended plane, where points on the diagonal $\Delta = \{(x,\ x) \in \mathbb{R}^2\}$ are added with infinite pultiplicity.



Example





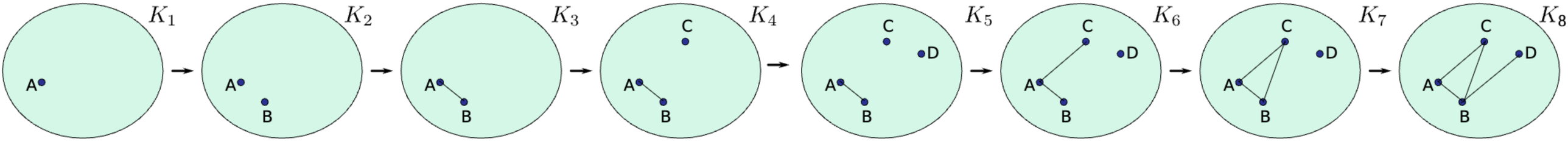


Determine $\mu_0^{i,\,j}$ for all pairs $i \le j$

$$\mu_p^{i,j} = \begin{cases} (\beta_p^{i,j-1} - \beta_p^{i,j}) - (\beta_p^{i-1,j-1} - \beta_p^{i-1,j}), & i < j < n+1, \\ \beta_p^{i,j} - \beta_p^{i-1,j}, & i < j = n+1. \end{cases}$$

$\mu_p^{2,3} = \{ (\beta^{2,2} - \beta^{2,3}) - (\beta^{1,2} - \beta^{1,3})$

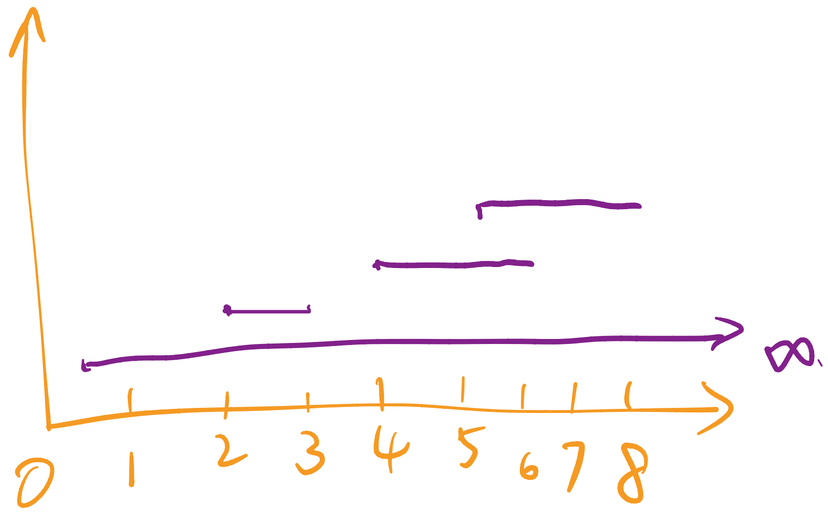
Example



Plot the 0-dimensional persistence diagram

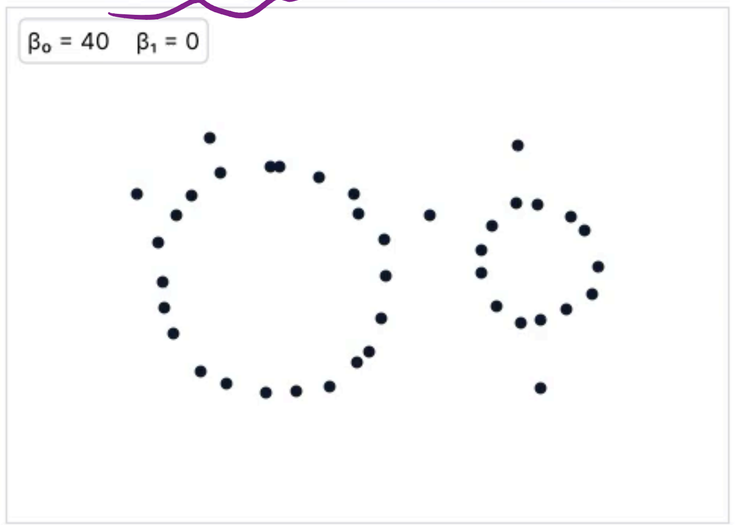
Barcode

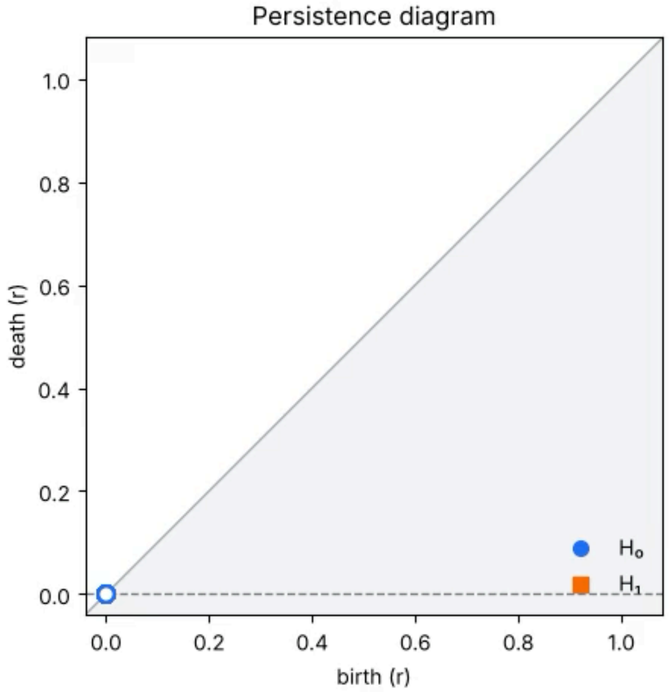
The pth barcode of a filtration is the multiset of half open intervals $[a_i,\ a_j)$ where each interval appears with multiplicity $(\mu_p^{i,j})$. Each interval is drawn as a horizontal bar recording the lifetime of an independent p-dimensional homological feature.

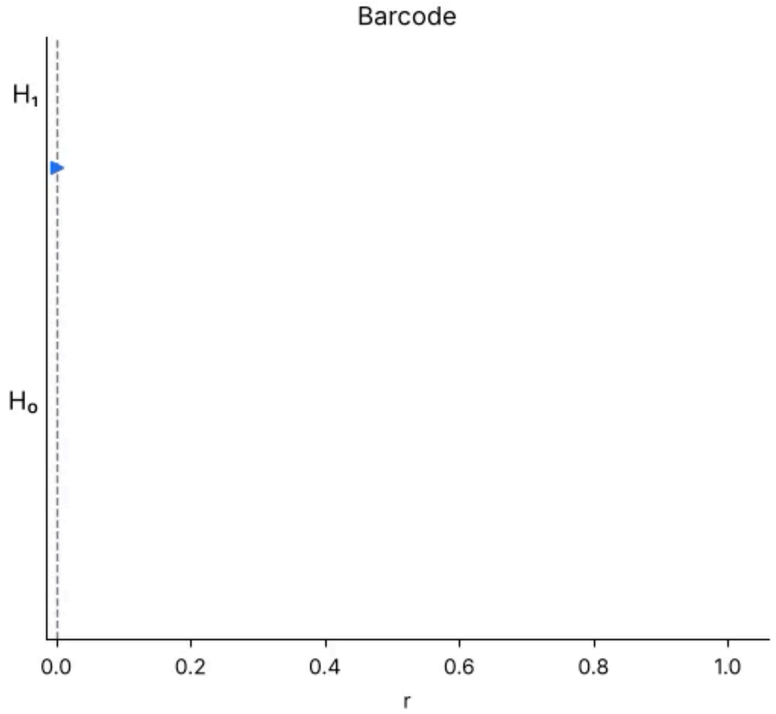


Putting it all together

filtration : radius







Takeaway Messages

- Track features through the filtration
- Count independent features, not every cycle
- The barcode is intrinsic; it's representative are not
- For applications: software handles the algebra

Homework presentation

- Determine the $\beta_0^{i,j}$ table, the $\mu_0^{i,j}$ table, the 0-dimensional persistence diagram, and the 0-dimensional barcodes for the filtrations below

