Simplicial Complexes

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Recall from last time

Topology	A rule telling us what it means for points to be locally near one anohter
Open set	Every point has some wiggle room around it that is in the set
Closed set	The set contains its limiting points
Metric topology	Distance defines “nearby”, balls give neighborhoods; unions of balls give open sets

Recall from last time

Maps	Continuous functions (nearby inputs don't jump to far-away outputs)
Connected	The space is one piece
Homeomorphism	Two spaces are the same topological shape. You can continuously transform one into the other and continuously undo it.
Embedding	Put one space inside another without changing its topology. No identifying/gluing different points together, and the topology on the image must agree with the original one.
Homotopy	A continuous movie of maps: g gradually changes into h, the parameter t is time.

Homotopy

Let $g : X \to U$ and $h : X \to U$ be maps.

A *homotopy* is a map $H : X \times [0,1] \to U$ such that $H(\cdot,0) = g$ and $H(\cdot,1) = f$

Two maps are homotopic if there is a homotopy connecting them.

Annulus Example

Annulus: $A = \{(\theta, r) \mid 1 \leq r \leq 2\}$. Circle: $\mathbb{S}^1 = \{(\theta, r) \mid r = 1\}$

$g : A \to A$ identity. $h : A \to A, h(\theta, r) = (\theta, 1)$.

Show $R(\theta, r, t) = (\theta, (1 - t)r + t)$ is a homotopy

Homotopy Equivalent

Two topological spaces T and U are *homotopy equivalent* if there exist maps $g : T \to U$ and $h : U \to T$ such that $h \circ g$ and $g \circ h$ are homotopic to the appropriate identity maps.

Ex. Show A is homotopy equivalent to $\mathbb{S}^1 \subset A$

Retract

Let T be a topological space, and let $U \subset T$ be a subspace.

A *retraction* r of T to U is a map $r : T \to U$ such that $r(x) = x$ for every $x \in U$

Ex. Annulus to circle

Deformation Retract

The space $U \subseteq T$ is a *deformation retract* of T if the identity map on T can be continuously deformed to a retraction with no motion of the points already in U. Specifically, there is a homotopy $R : T \times [0,1] \rightarrow T$ such that

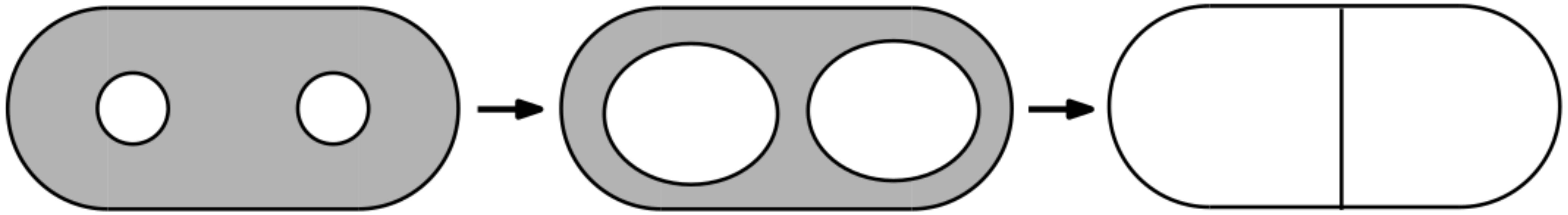
- $R(\cdot,0)$ is the identity map on T
- $R(\cdot,1)$ is a retraction of T to U
- $R(x,t) = x$ for every $x \in U$ and every $t \in [0,1]$

Annulus Example

$R(\theta, r, t) = (\theta, (1 - t)r + t)$. Check that R satisfies the three properties to be a deformation retract.

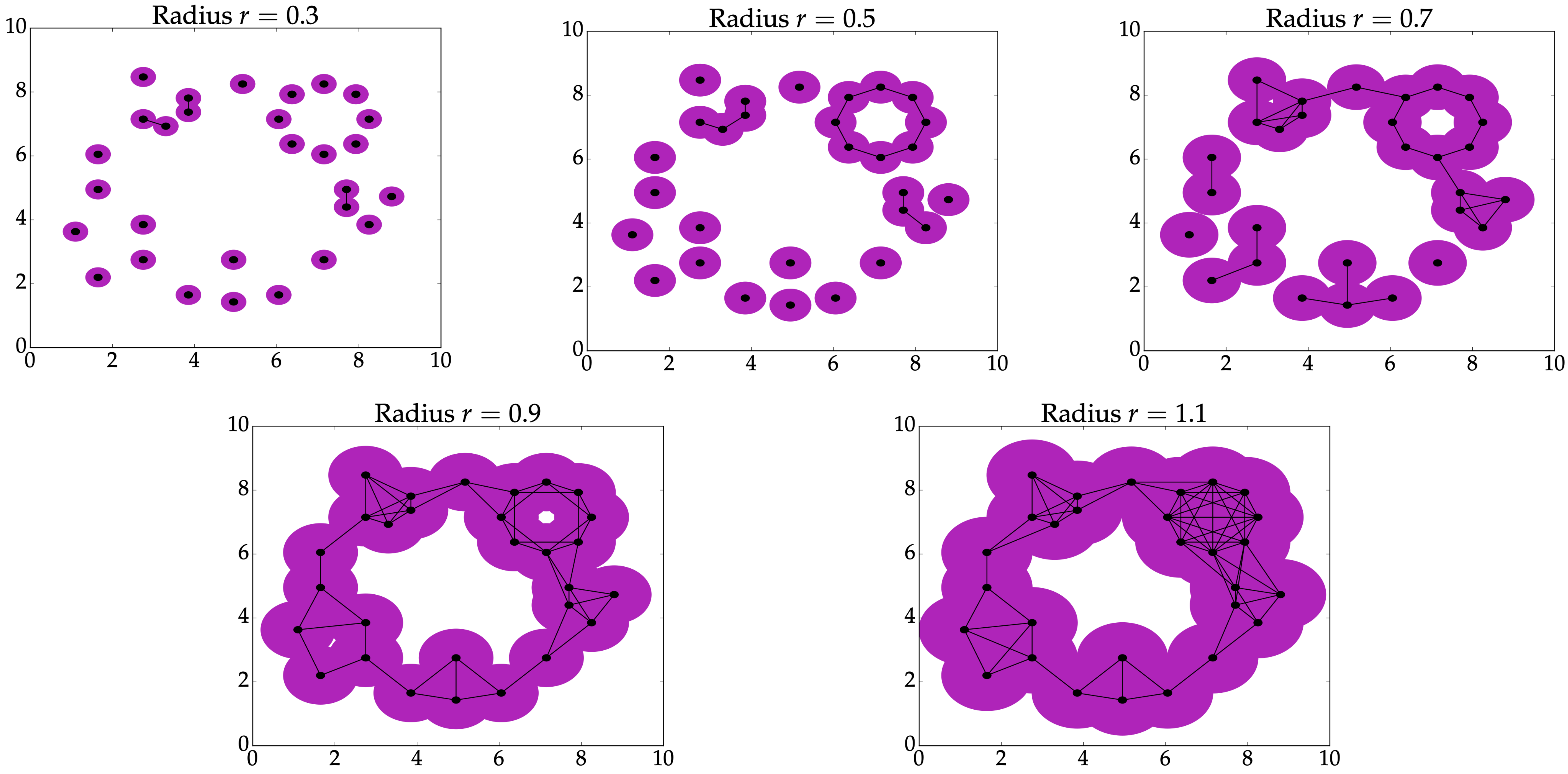
Why do I care?

If U is a deformation retract of T, then T and U are homotopy equivalent.



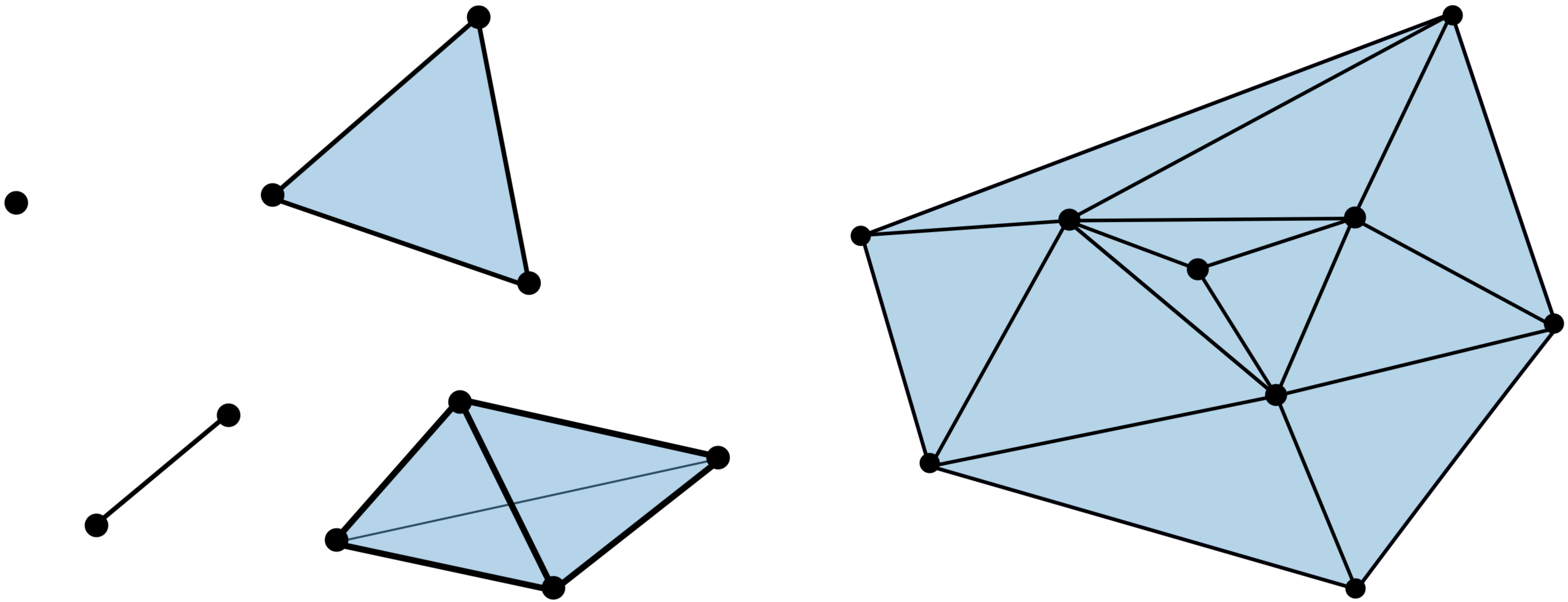
Ex. The letter A to O

Goal: what we want to study



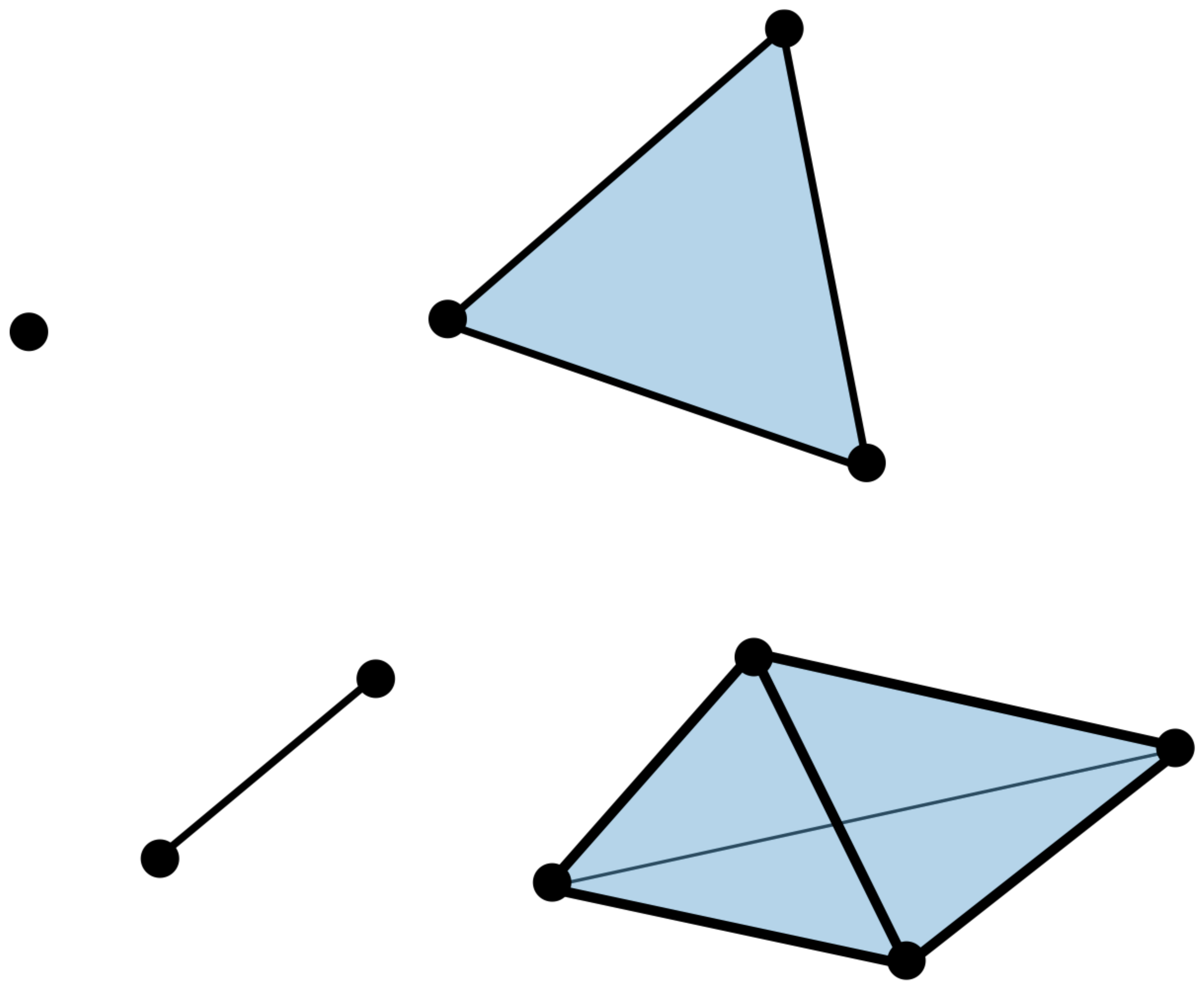
Simplicial Complexes

Idea: turn topology into combinatorics using mathematical “legos”



Building Blocks

- 0-simplex: a point (vertex)
- 1-simplex: a line segment (edge) connection two points
- 2-simplex: a triangle (face) filled in between three points
- 3-simplex: a tetrahedron (solid pyramid) filled in between four points
- Higher dimensions: the generalization of a triangle to that dimension

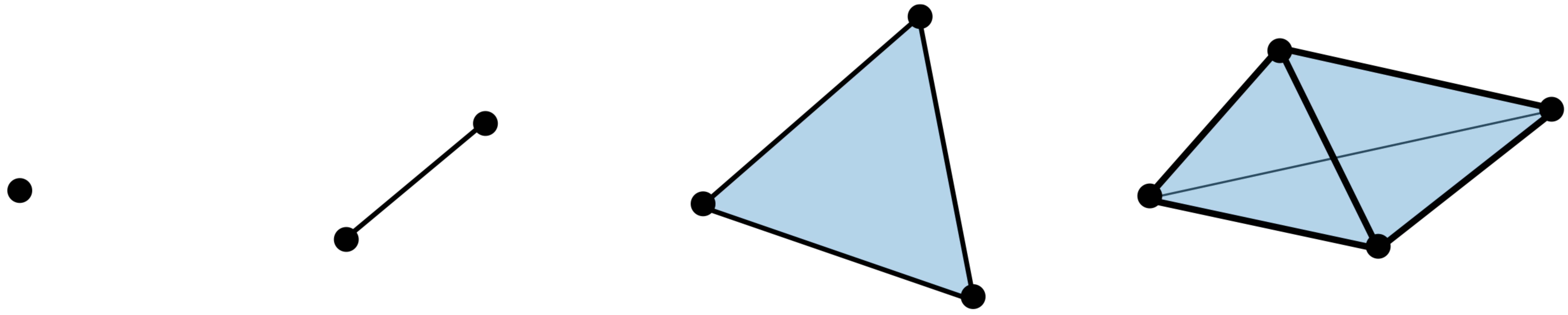


Gluring Rules

- Face Containment: if a simplex is in the complex, then its subsimplex must be too
  - Given a solid triangle, its three edges and three points must also be part of the complex
- Intersection: If two shapes intersect, their intersection must also be part of the complex

Simplicial Complex: Definition

An *n-simplex* in $\mathbb{R}^d$ is the convex hull of $n+1$ affinely independent points in $\mathbb{R}^d$.



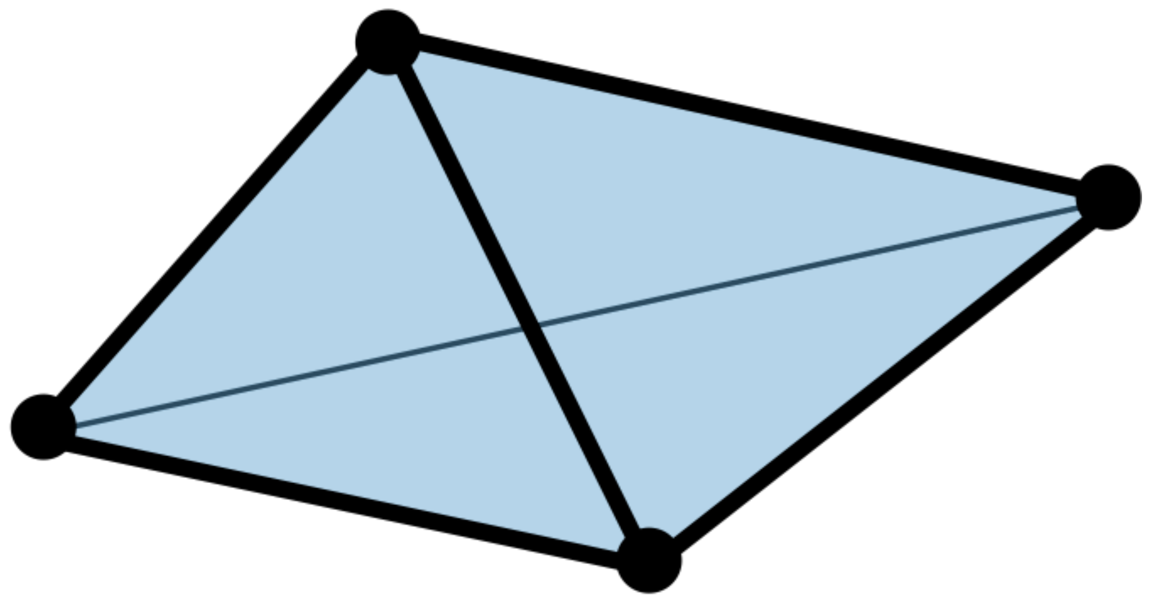
Face: convex hull of subsett of vertices

$\emptyset$ is also face

A face is proper if it isn't the simplex itself

Interior and Boundary

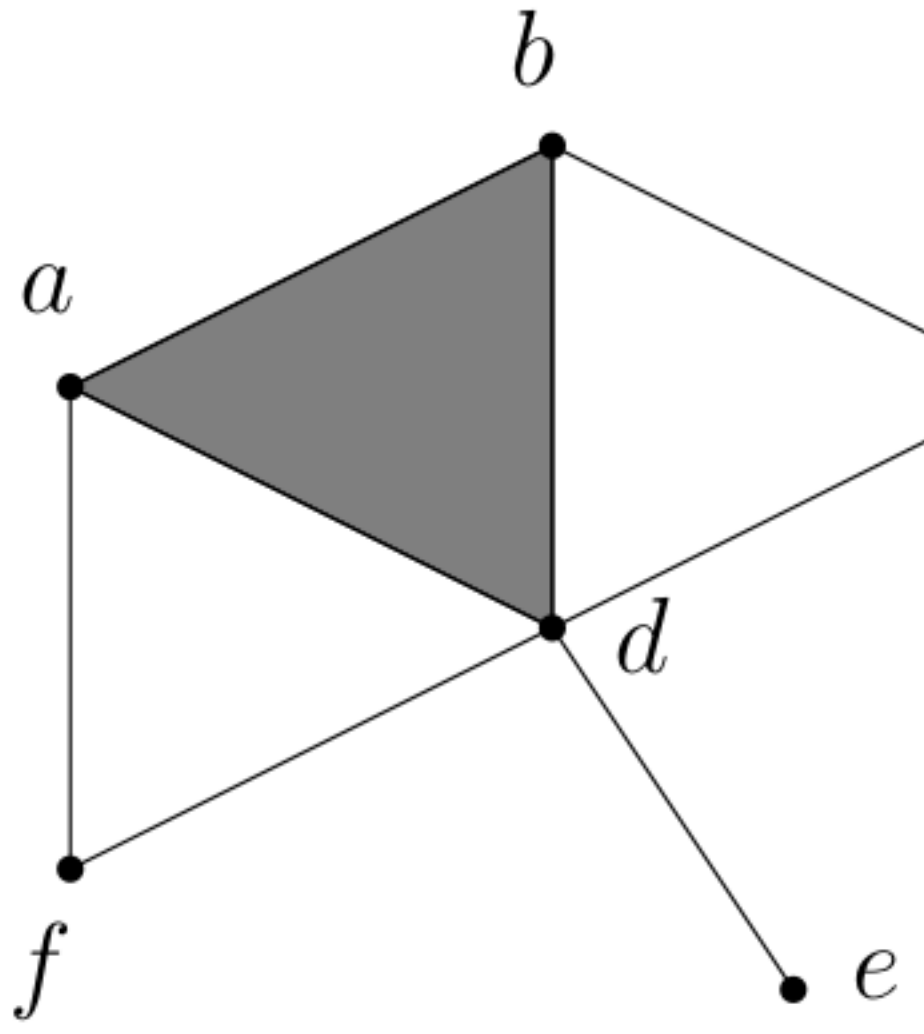
- Let σ be an n-simplex, the union of the proper faces is the *boundary* of σ, $\text{Bd}(\sigma)$.
- The *interior* of σ is $\sigma - \text{Bd}(\sigma)$. This is sometimes called the open simplex



Geometric Simplicial Complex

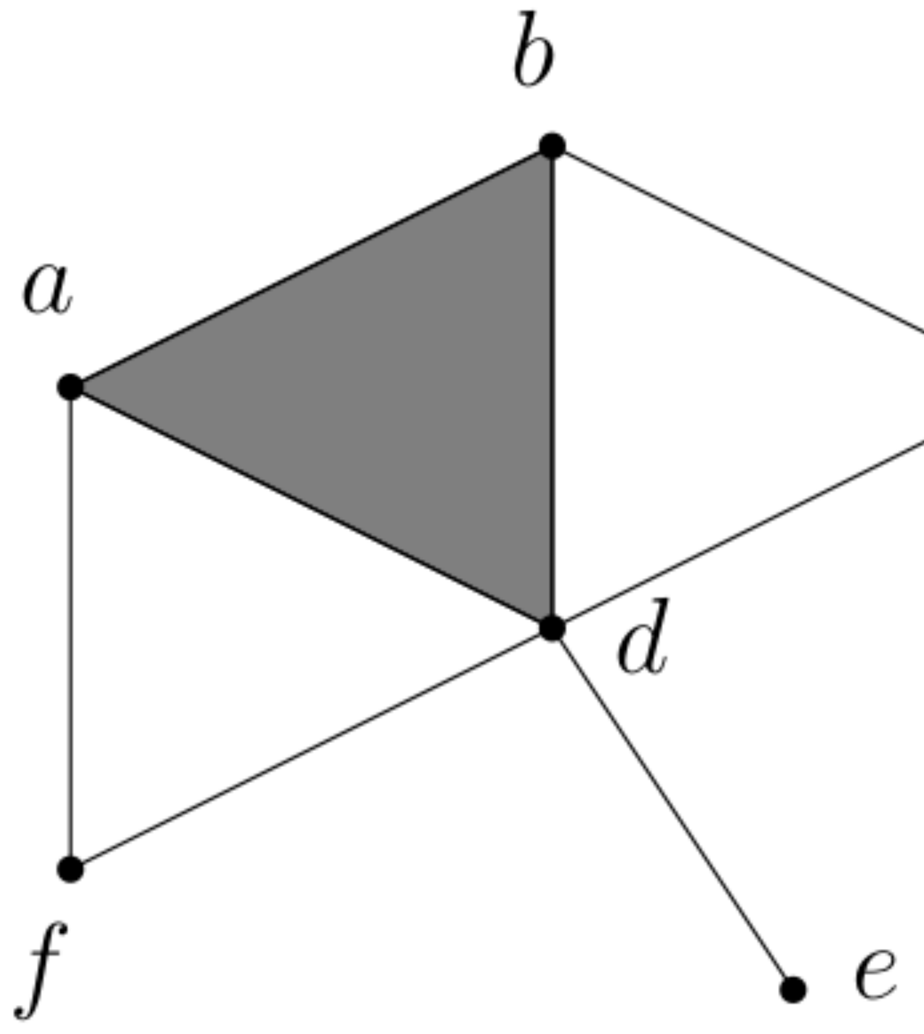
A <u>*geometric simplicial comlex*</u> K in $\mathbb{R}^d$ is a finite collection of simplices in $\mathbb{R}^d$ such that

- Every face of a simplex of K is in K
- The intersection of any two simplices of K is a face of each.



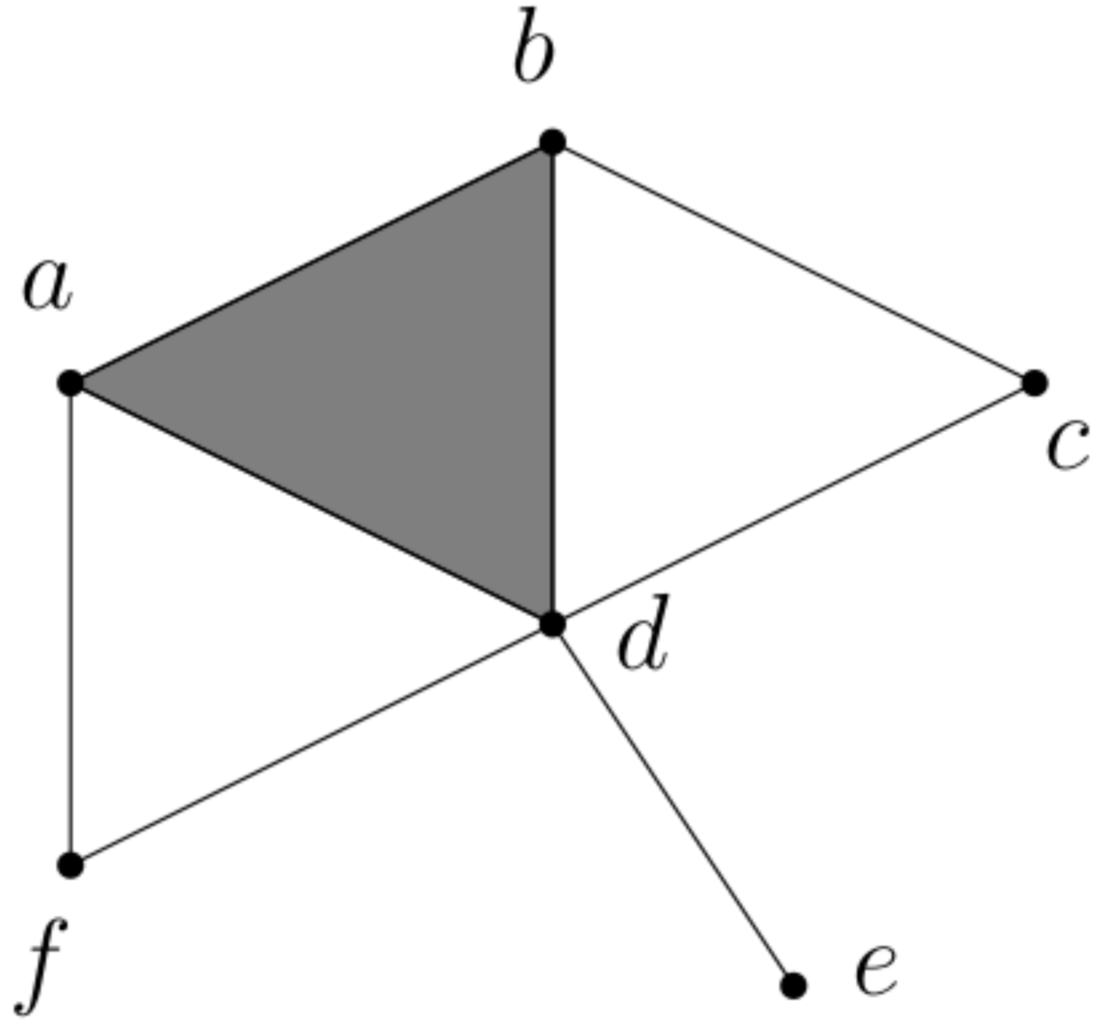
Abstract Simplicial Complex

An <u>*abstract simplicial comlex*</u> $\mathscr{K}$ is a finite collection of finite non-empty sets of V such that $\alpha \in \mathscr{K}$ and $\beta \subseteq \alpha$ implies $\beta \in \mathscr{K}$.



Geometric Realizations

Geometric simplicial complex K is <u>*geometric realization*</u> of abstract simplicial comlex $\mathscr{K}$ if there is an embedding $f : V(\mathscr{K}) \to \mathbb{R}^d$ that takes every abstract n-simplex $\{v_0, \ldots, v_n\}$ in $\mathscr{K}$ to the geometric k-simplex that is the convex hull of $f(v_0), \ldots, f(v_n)$.



Theorem: Every n-dimensional simplicial complex has a geometric realization in $\mathbb{R}^{2n+1}$

Underlying Space

$|K|$ is the *underlying space* of K, $|K| = \bigcup_{\sigma \in K} \sigma$

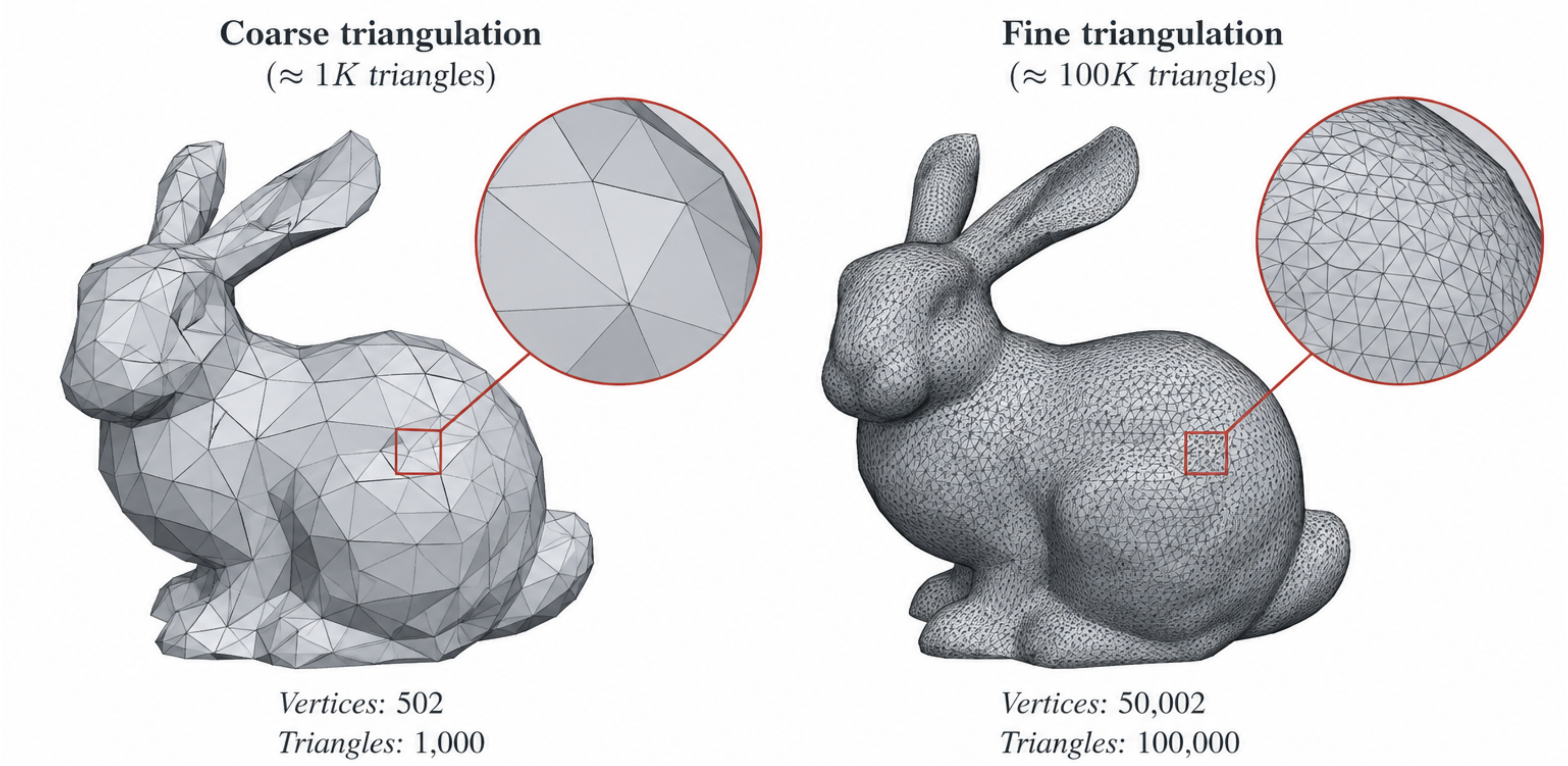
Ex. $K = \{\{v_0\}, \{v_1\}, \{v_2\}, \{v_0, v_1\}, \{v_1, v_2\}, \{v_0, v_2\}, \{v_0, v_1, v_2\}\}$

Triangulations

A simplicial complex K is a *triangulation* of a topological space T if $|K|$ is homeomorphic to T.

Note: Not all spaces are triangulable (topologists's sine curve);

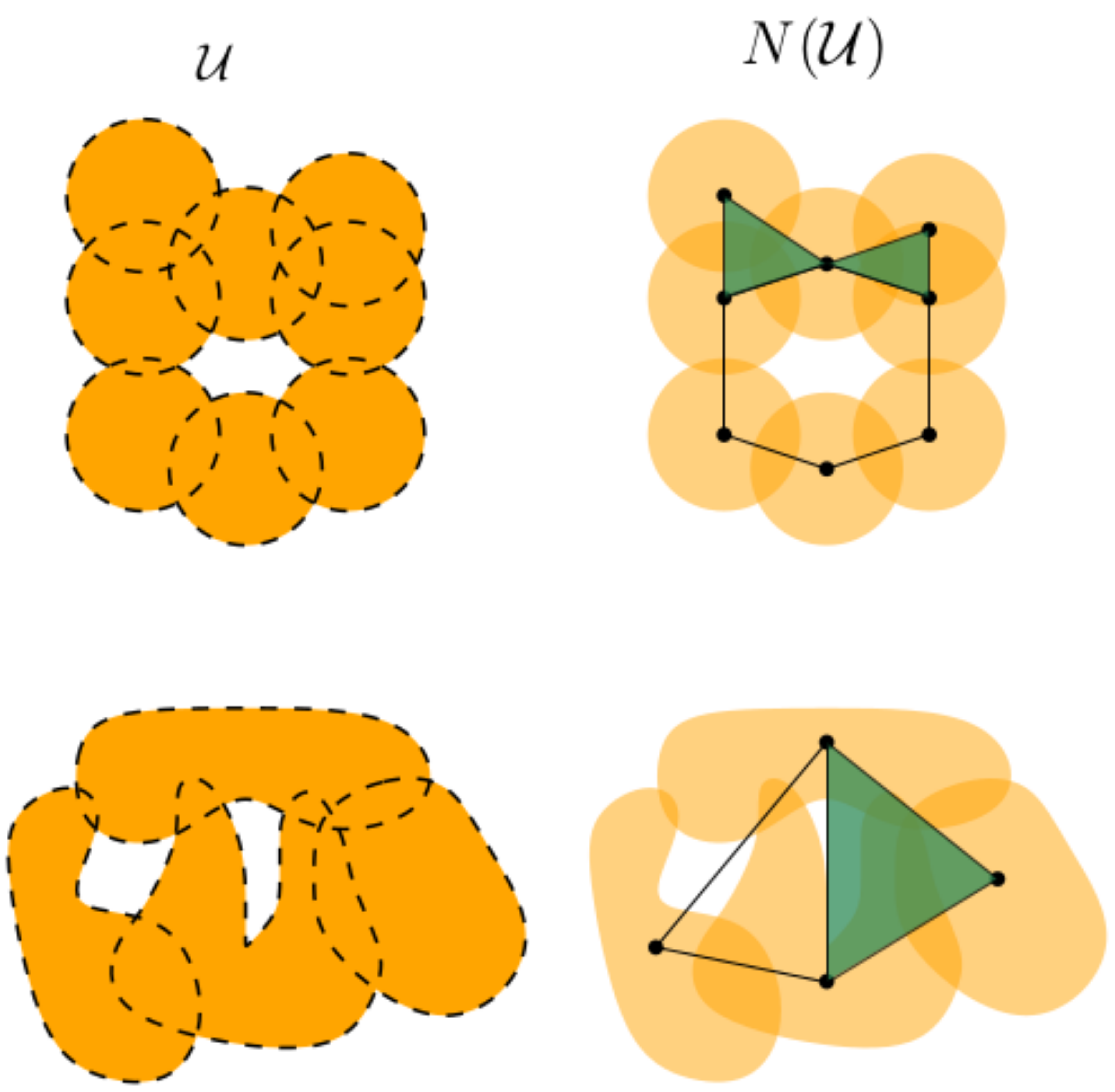
A triangulable space has infinitely many triangulations.



Nerve

Given a finite collection of sets $\mathscr{F}$, the <u>*nerve*</u> is

$$\mathrm{Nrv}(\mathscr{U}) = \{X \subseteq \mathscr{F} \mid \bigcap_{U \in X} U \neq \emptyset\}$$

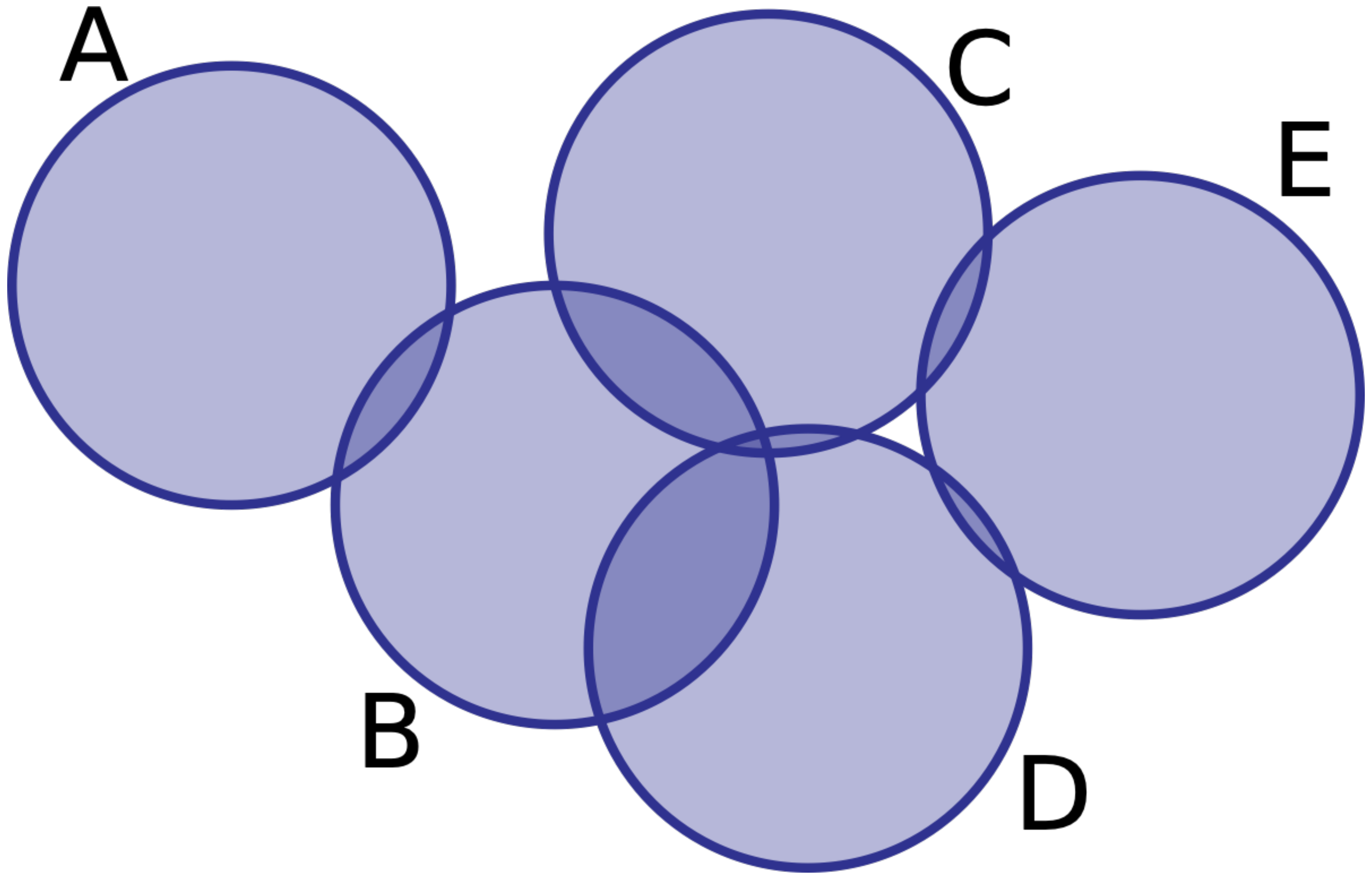


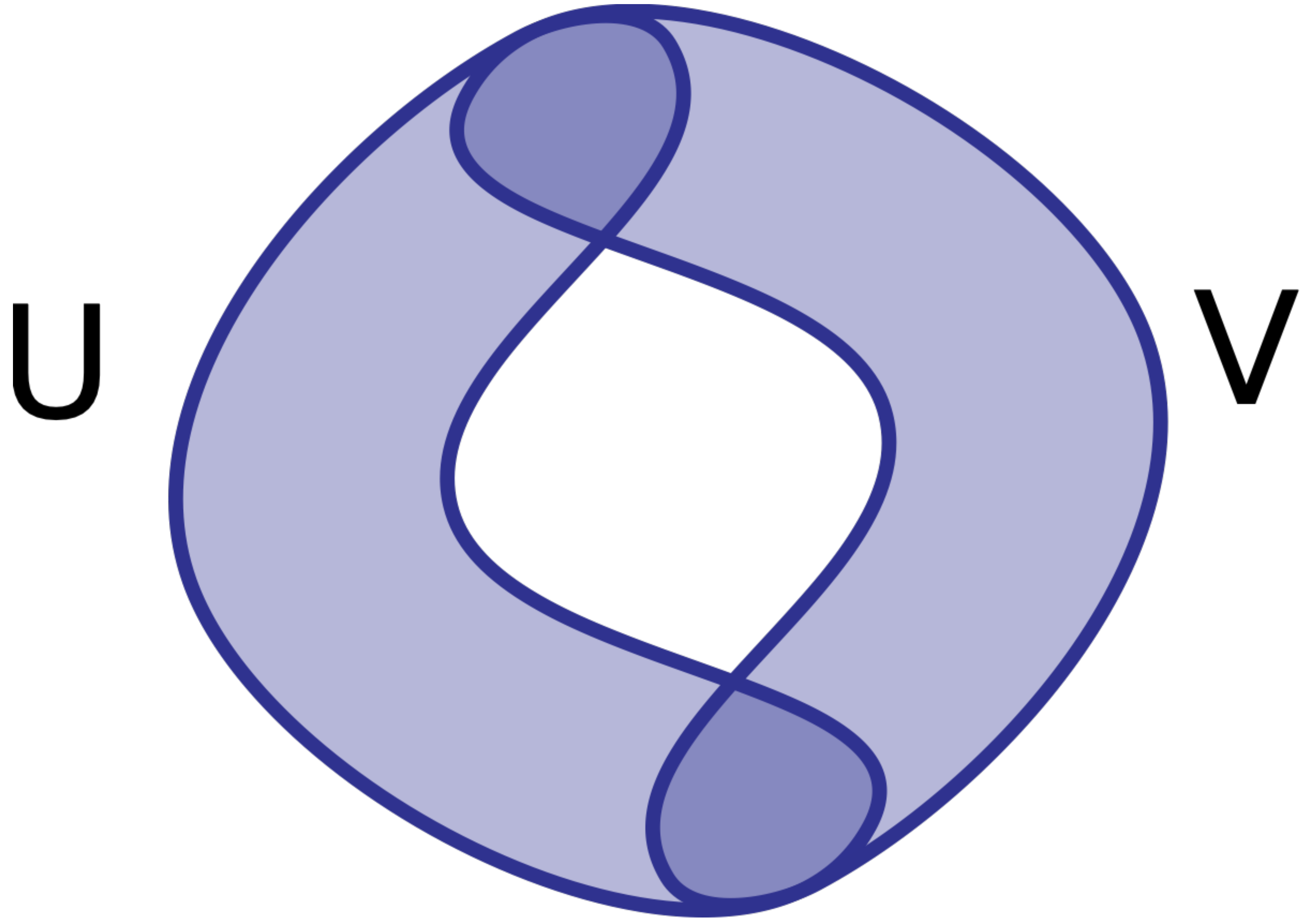
Check: This is actually an abstract simplicial complex!

Recall: K is a simplicial complex if $\sigma \in K$ and $\tau \leq \sigma$ implied $\tau \in K$

$$\text{Nrv}(\mathscr{U}) = \{X \subseteq \mathscr{F} \mid \bigcap_{U \in X} U \neq \emptyset\}$$

Examples





Čech and Rips Complexes

Point Cloud

A *point cloud* is a finite collection of points in a metric space (M, d)

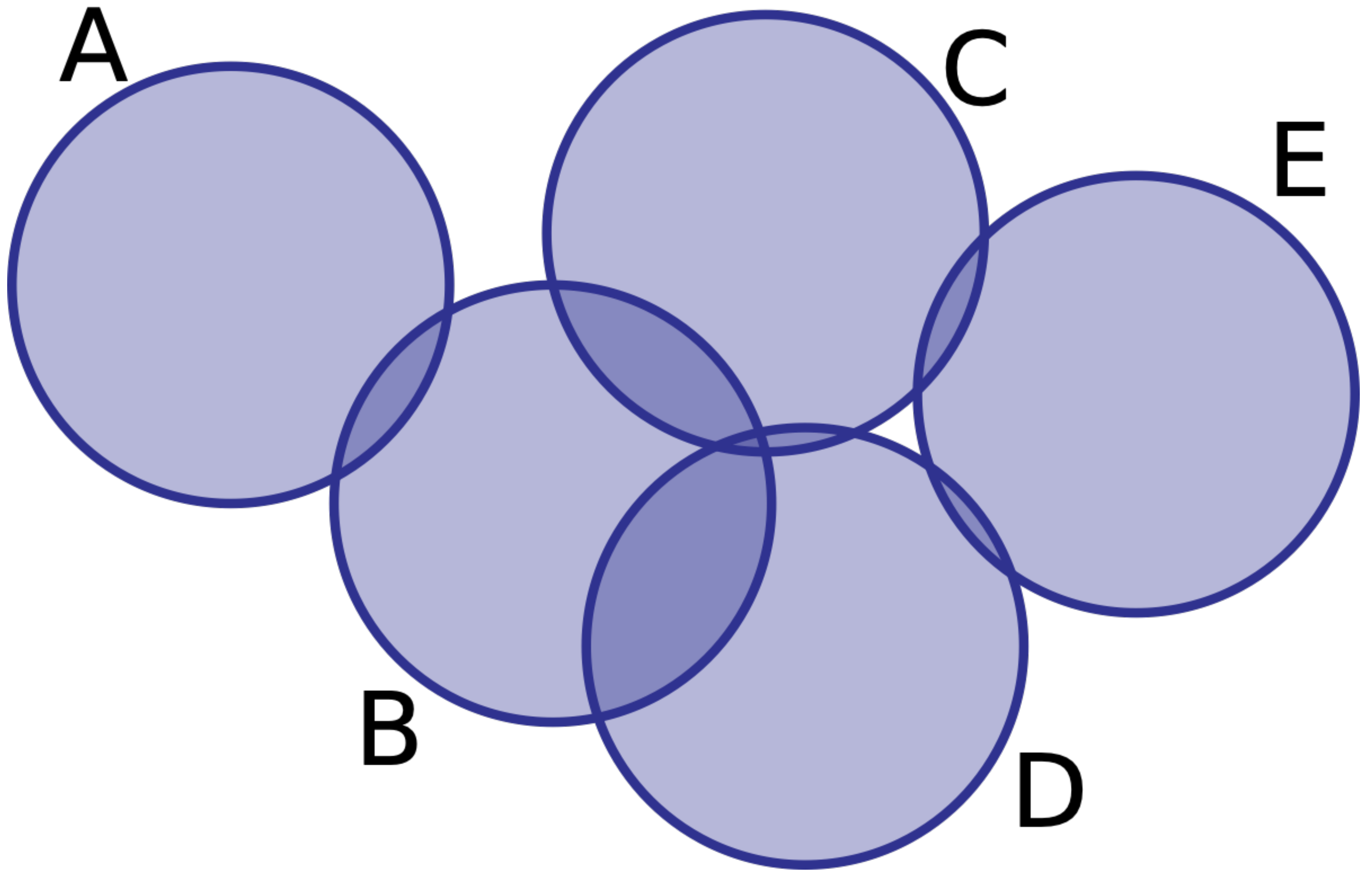
Ball

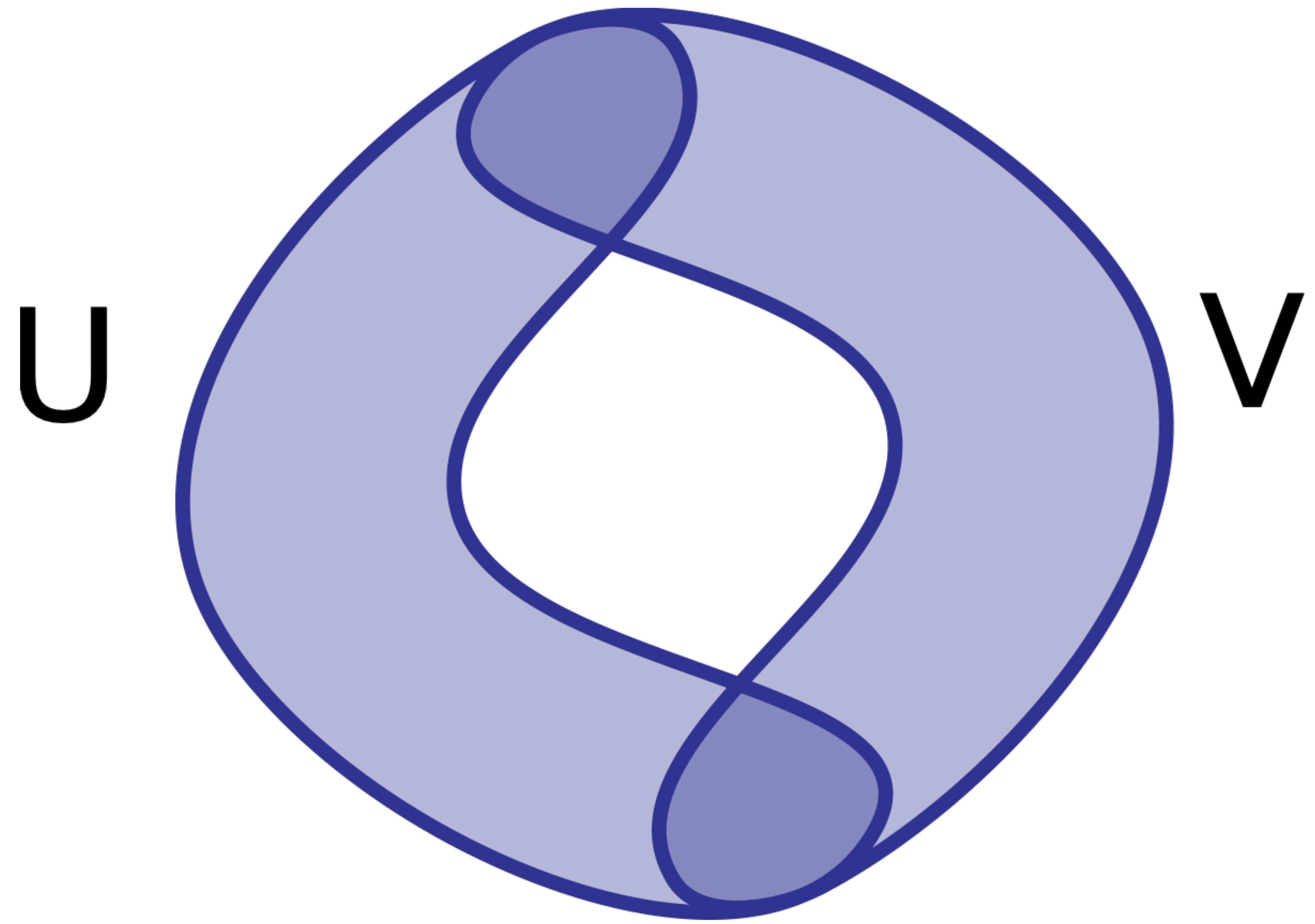
$B_0(x, r) = \{y \in M \mid d(x, y) < r\}$

$B(x, r) = \{y \in M \mid d(x, y) \leq r\}$

The Nerve Lemma

Given a finite cover $\mathscr{U}$ (open or closed) of a metric space M, the underlying space $|N(\mathscr{U})|$ is homotopy equivalent to M if every non-empty intersection $\bigcap_{i=0}^{k} U_{\alpha_i}$ of cover elemets is homotopy equivalent to a point (contractible)





Čech Complex

Let $P \subset (M, d)$ be a finite point cloud. Fix $r \geq 0$. The Čech complex is

$$\check{\mathrm{C}}^r(P) = \{\alpha \subseteq P \mid \bigcap_{x \in \alpha} B(x, r) \neq \emptyset\} = \mathrm{Nrv}(\{B(x, r)\}_{x \in P})$$

Warning

The Čech complex is an abstract simplicial complex. The obvious map into $\mathbb{R}^d$ doesn’t necessarily get you a geometric simplicial complex!

Rips Complex

Given $P \subseteq (M, d)$, the Vietoris-Rips complex is

$\mathrm{VR}^r(P) = \{\alpha \subseteq P \mid d(x_i, x_j) \leq 2r$ for all $x_i, x_j \in \alpha\}$

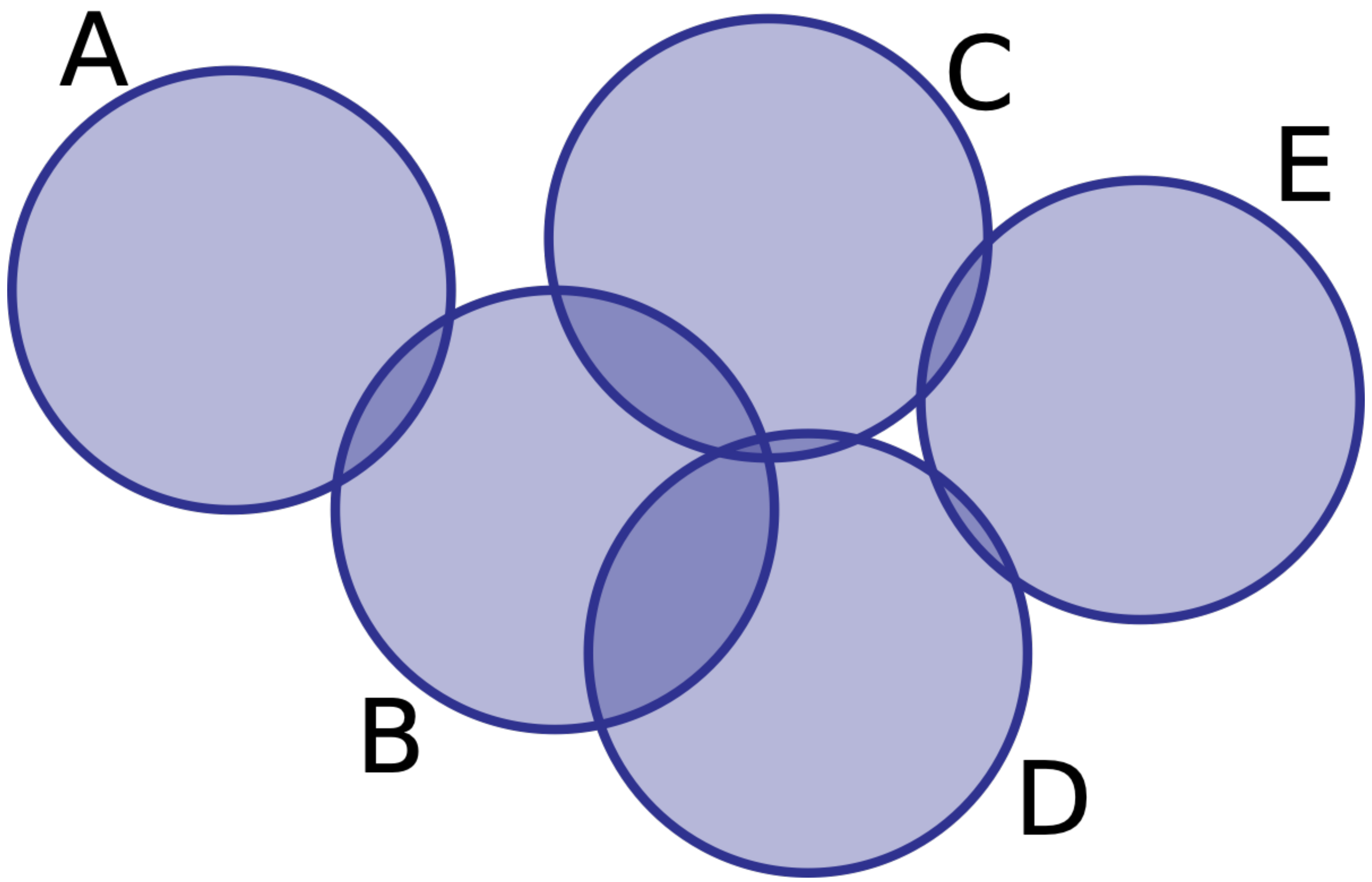
Equilateral Triangle Example

Rips-Čech Lemma

Given point cloud $P \subset (M, d)$ and $r > 0$, $\check{C}^r(P) \subseteq \mathrm{VR}^r(P) \subseteq \check{C}^{2r}(P)$

Warning: Radius vs Diameter

$$\mathrm{VR}^r(P) = \{\alpha \subseteq P \mid d(x_i, x_j) \leq 2r \text{ for all } x_i, x_j \in \alpha\}$$



Alpha Complex

Voronoi Diagram

Given point cloud $P \subseteq \mathbb{R}^d$. The Voronoi cell of $u \in P$ is

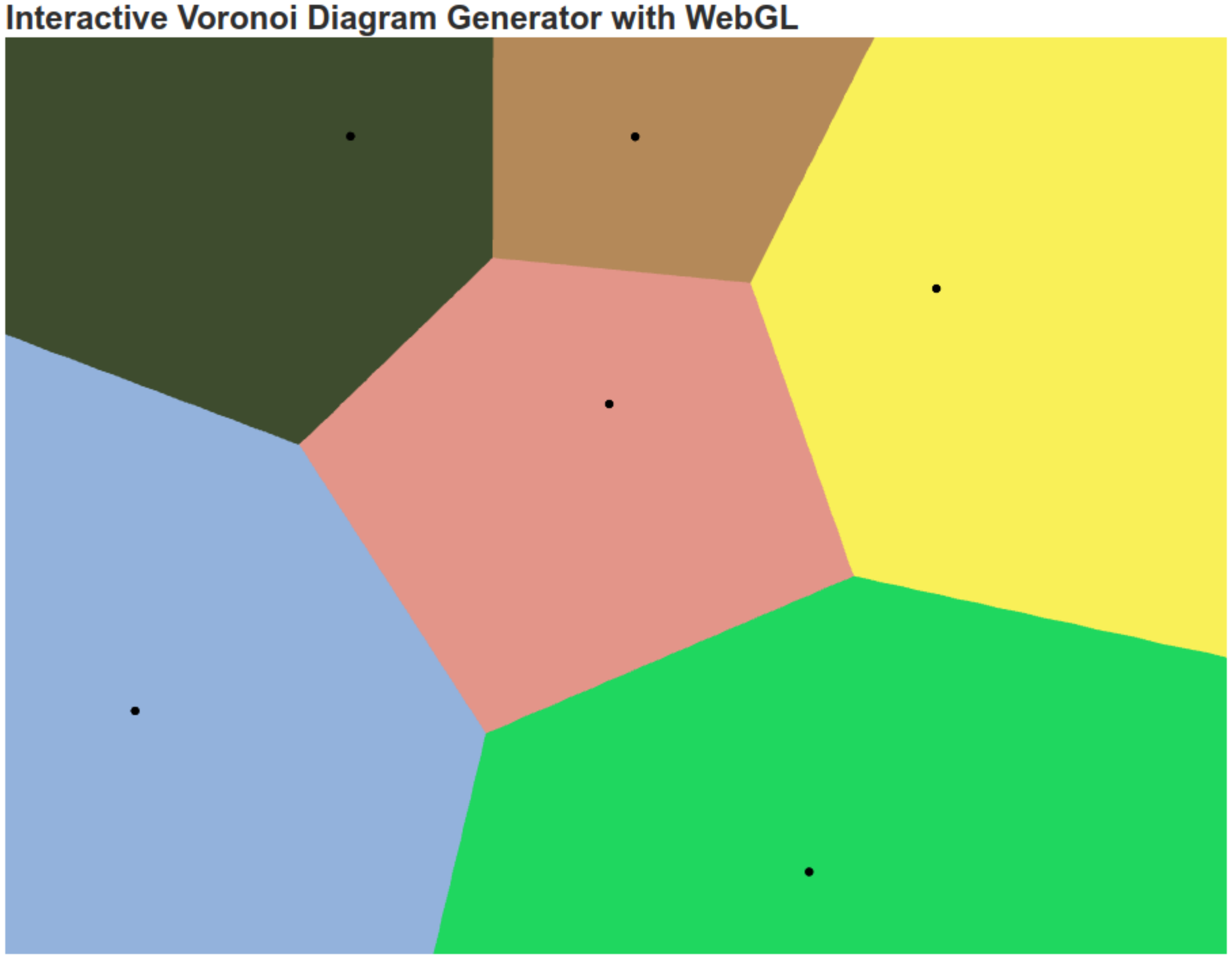
$V_u = \{x \in \mathbb{R}^d \mid \|x - u\| \leq \|x - v\|, v \in P\}$

The Voronoi diagram is the collection of Voronoi cells $\mathrm{Vor}(P) = \{V_u \mid u \in P\}$

Delaunay Triangulation

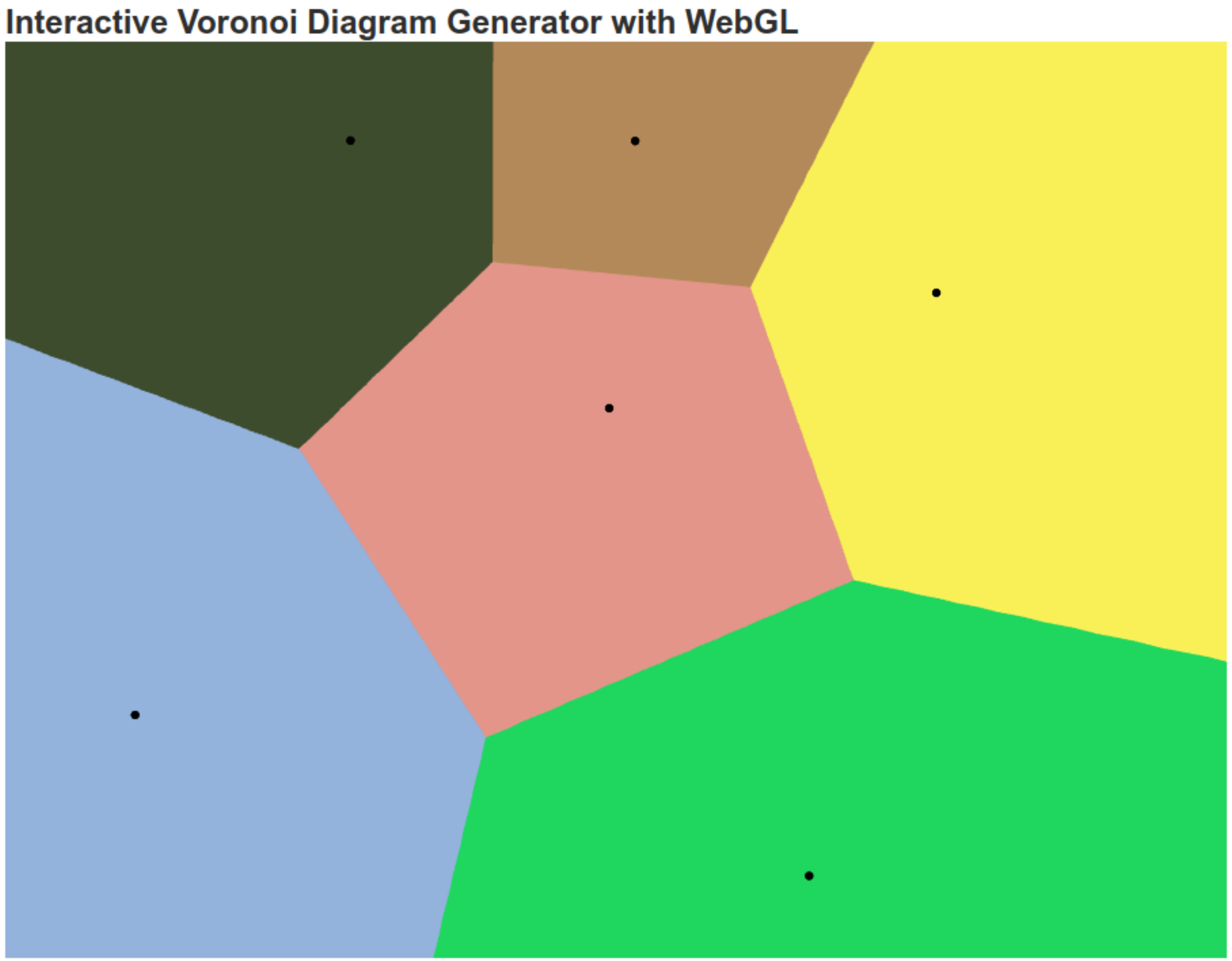
The Delaunay complex of point cloud $P \subseteq \mathbb{R}^d$ is the nerve of the Voronoi Diagram

$$\mathrm{Del}(P) = \{\sigma \subseteq P \mid \bigcap_{u \in P} V_u \neq \emptyset\}$$

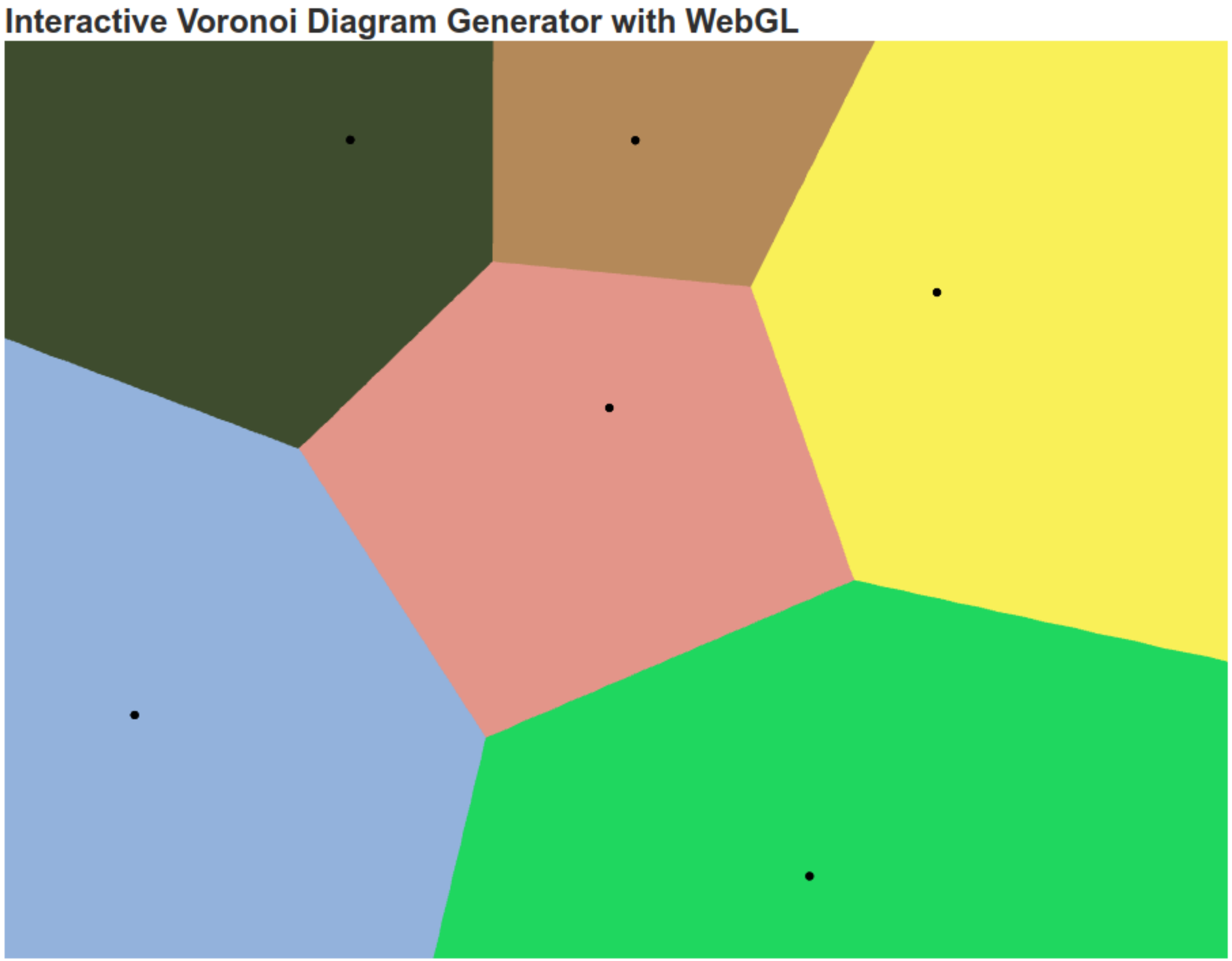


Properties

- Delauney is an abstract simplicial complex
- If we have points in general position, the obvious embedding gives a geometric simplicial complex
- Delauney is FIXED (has nothing to do with a radius or diameter parameter)



Alpha Complex



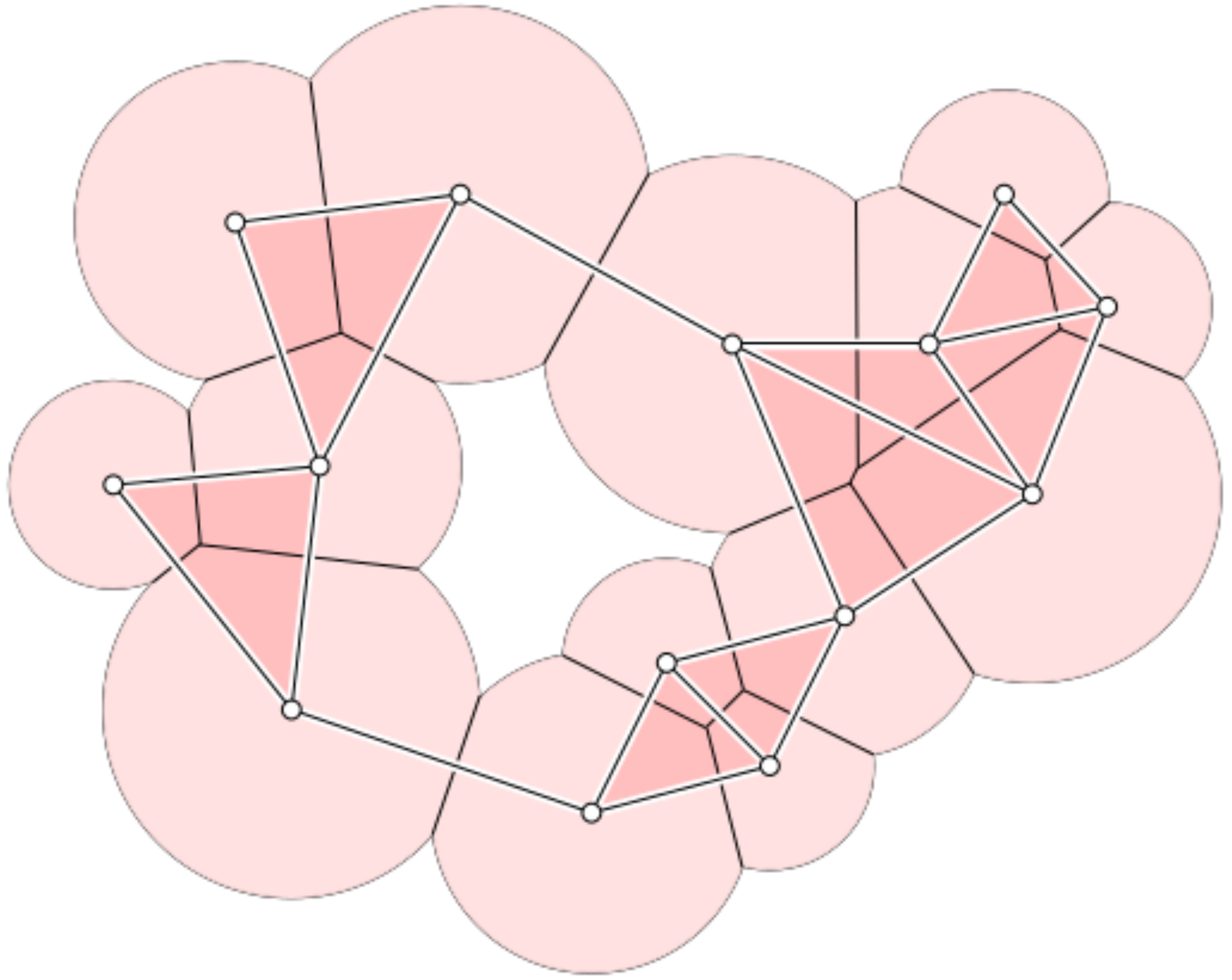
Alpha Complex

The Delaunay complex of point cloud $P \subseteq \mathbb{R}^d$ is the nerve of the Voronoi Diagram

$\mathrm{Del}_P^\alpha = \{x \in B(p, \alpha) \mid d(x, p) \leq d(x, q)$ for all $q \in P\} = B(p, r) \cap V_p$

The alpha comlex for point cloud P with radius $r \geq 0$ is the nerve

$\mathrm{Alpha}(r) = \mathrm{Del}_P^\alpha = \mathrm{Nrv}(\{D_P^\alpha \mid p \in P\})$



Properties

- Alpha(r) $\subseteq$ Delaunay
- Alpha(r) $\subseteq$ Č(r)
- Alpha(r) has the same homotopy type as the union of balls of radius r